Generic Function Operations

We will now consider an approach in which the generalized function is notated as (f,φ) not only correct, but also quite effective.

The essence of the technique is this: we get a symbolic form of writing some operation with a generalized function for *regular* case (when the use of the integral is permissible), and then (after making sure of the linearity and continuity of the result of the operation) we use this form of notation for *singular* case, taking it as a definition.

Multiplying an ordinary function by a generalized one

We use this scheme to determine in D' "multiplication by function" operations.

Let $g(x)$ – infinitely differentiable ordinary function. What can we take for $g(x)f(x)$, If $f(x)\in D'$?

In the regular case we have $(g(x)f,\,\varphi)=\int\limits_{-\infty}^{+\infty}g(x)f(x)\varphi(x)\,dx=(f,\,g(x)\varphi)$. For this functionality $g(x)f$ can be accepted $(g(x)f,\,\varphi)=(f,\,g(x)\varphi)$.

Note that if $\varphi(x)$ – main function, then it will be the main and $g(x)\cdot\varphi(x)$. Check the linearity and continuity of the new functionality in this definition yourself.

Example 1. Find a generic function $y(x)$, which is a solution to the equation $xy(x)=0$.

Solution: We have in D' $p(x)\delta(x)=p(0)\delta(x)$ for any infinitely differentiable function $p(x)$. Really,

$$\big(p(x)\delta(x),\varphi(x)\big)=\big(\delta(x),\,p(x)\varphi(x)\big)=p(0)\varphi(0)=.\big(p(0)\delta(x),\varphi(x)\big)$$

For $p(x)=x$ obviously $p(0)=0$, therefore the solution to this equation is, for example, . $y(x)=C\delta(x)$

Differentiation of generalized functions

Using the same technology it is possible to determine *derivative* for a generic function. This definition has the form

$$(f', \varphi) = -(f, \varphi') \quad . \qquad (1)$$

Indeed, for *regular* generalized function $f(x)$, which is generated by the ordinary one. continuously differentiable function, according to the rule of integration by parts, we have, due to finiteness, $\varphi(x)$ And $\varphi'(x)$

$$f' = (f', \varphi) = \int\limits_{-\infty}^{+\infty} f'(x)\varphi(x)\,dx = f(x)\varphi(x)\Big|_{-\infty}^{+\infty} - \int\limits_{-\infty}^{+\infty} f(x)\varphi'(x)\,dx = -(f, \varphi') ,$$

which gives grounds to accept formula (1) as the definition of the derivative of a generalized function.

From formula (1) it follows that

- *each* generalized function has derivative *any* order,

- operation of differentiation of a generalized function *linear*:

$$((\lambda_1 f_1 + \lambda_2 f_2)', \varphi) = \lambda_1 (f_1', \varphi) + \lambda_2 (f_2', \varphi).$$

For generalized functions the analogue is valid *Leibniz formulas*. Consider, for example, the case $n=1$.

Let $f(x)$ is an arbitrary generalized function, and $g(x)$ − a regular generalized function generated by an infinitely differentiable ordinary function. Then

$$(f\cdot g)'=((f\cdot g)',\varphi)=-((f\cdot g),\varphi')=-(f,g\varphi')=$$

$$=-(f,(g\cdot\varphi)'-g'\varphi)=(f,g'\varphi)-(f,(g\cdot\varphi)')=$$

$$=(fg',\varphi)+(f',g\varphi)=(fg',\varphi)+(f'g,\varphi)=$$

$$=(f'g,\varphi)+(fg',\varphi)=(f'g+fg',\varphi)=f'g+fg'.$$

Example 2. Find f' And f'' for a generalized function generated by an ordinary function

$$f(x)=\begin{cases}\alpha x \text{ при } x\le 0,\\ \beta x \text{ при } x>0.\end{cases}$$

Solution: 1. We have

$$f'=-(f,\varphi')=-\int_{-\infty}^{+\infty} f(x)\varphi'(x)\,dx=-\alpha\int_{-\infty}^{0} x\varphi'(x)\,dx-\beta\int_{0}^{+\infty} x\varphi'(x)\,dx=$$

$$=-\alpha\left(x\cdot\varphi(x)\Big|_{-\infty}^{0}-\int_{-\infty}^{0}\varphi(x)\,dx\right)-\beta\left(x\cdot\varphi(x)\Big|_{0}^{+\infty}-\int_{0}^{+\infty}\varphi(x)\,dx\right)=$$

$$=\int_{-\infty}^{+\infty}\left(\alpha+(\beta-\alpha)\theta(x)\right)\varphi(x)\,dx=\int_{-\infty}^{+\infty} f'(x)\varphi(x)\,dx\,,$$

Where *Heaviside function*
$$\theta(x)=\begin{cases}0 & \text{при } x<0,\\ \frac{1}{2} & \text{при } x=0,\\ 1 & \text{при } x>0.\end{cases}$$

As a result,
$$f'(x)=\begin{cases}\alpha & \text{при } x<0,\\ \frac{\alpha+\beta}{2} & \text{при } x=0,\\ \beta & \text{при } x>0.\end{cases}$$
Or $f'(x)=\alpha+(\beta-\alpha)\theta(x)$.

2. For the second derivative, using similar reasoning, we obtain

$$f''=-(f',\varphi')=(f,\varphi'')=-\int\limits_{-\infty}^{+\infty} f'(x)\varphi'(x)\,dx=-\alpha\int\limits_{-\infty}^{0}\varphi'(x)\,dx-\beta\int\limits_{0}^{+\infty}\varphi'(x)\,dx=$$

$$=-\alpha\varphi(x)\Big|_{-\infty}^{0}-\beta\varphi(x)\Big|_{0}^{+\infty}=(\beta-\alpha)\varphi(0)=(\beta-\alpha)\delta(x).$$

3. Considering that when $\alpha=-1$ And $\beta=1$ we have $f(x)=|x|$, then in space D' the equalities will be true $|x|'=\operatorname{sgn}x$ And $|x|''=2\delta(x)$.

Problem 2 illustrates the following rules for differentiating regular generalized functions with discontinuities of the 1st kind, both for the functions themselves and for their derivatives.

Suppose that the generating function is continuous, but its derivative has a discontinuity of the first kind at the point x_0 with a "jump" of value equal to *A*. Then the derivative of the generalized function will be equal to the derivative of the generating function with an addition of the form $A\theta(x-x_0)$.

If a jump of the first kind at the point x_0 quantities *A* is present in the generating function, then its generalized derivative has the term $A\delta(x-x_0)$.

The formula is often useful for solving problems

$$p(x)\delta'(x) = p(0)\delta'(x) - p'(0)\delta(x)\ .$$

For example: $x\delta'(x) = -\delta(x)$.

Example 3. Find in D' second derivative for a regular function $f(x) = |x| \sin x$.

Solution: According to Leibniz's formula we have

$$f'' = (|x| \sin x)'' = |x|'' \sin x + 2|x|'(\sin x)' + |x|(\sin x)''.$$

Since $|x|' = \operatorname{sgn} x$ и $|x|'' = 2\delta(x),$ then we get

$$f'' = 2\delta(x)\sin x + 2\operatorname{sgn} x \cdot \cos x - |x| \sin x.$$

This formula can be simplified using the method for solving Problem 1.

Indeed, in D' $\delta(x)\sin x = 0$, because

$$\big(\delta(x)\sin x, \varphi(x)\big) = \big(\delta(x), \varphi(x)\sin x\big) = \varphi(0)\sin 0 = 0.$$

Therefore, finally

$$f'' = 2\operatorname{sgn} x \cdot \cos x - |x| \sin x.$$