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Using Feedback Functions in Linear Programming Problems

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Abstract—A scheme for solving linear programming problems based on the use of auxiliary functions that implement feedback in the system of constraints imposed on the variables to be found and on the Lagrange multipliers is proposed. The validity of the proposed approach is proved.

Keywords: linear program, penalty function method, feedback functions, modified Lagrangian function, regularization

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1. INTRODUCTION

We consider a version of the method of asymptotic estimates (which is a variation of perturbation method) for solving the standard linear program in which the

$$\text{objective function} \quad F(x) \tag{1.1}$$

should be maximized over $x \in E^n$ with the coordinate representation $\|x\| = \|\xi_1, \xi_2, \ldots, \xi_n\|^{\mathrm{T}}$

$$\text{subject to the constraints} \quad \xi_j \geq 0 \quad \forall j = [1, n], \quad f_i(x) \leq 0 \quad \forall i = [1, m], \tag{1.2}$$

where each component of the function $F(x)$, $f_i(x)$ $\forall i = [1, m]$ is linear in all its arguments.

2. OUTLINE OF THE METHOD OF FEEDBACKS FOR LINEAR PROGRAMS

First, consider the use of the proposed approach for solving a pair of mutually dual linear programs written in symmetric form.

Let the vectors $x \in E^n$ and $\Lambda \in E^m$ with the coordinate columns $\|x\| = \|\xi_1\xi_2\ldots\xi_n\|^{\mathrm{T}}$ and $\|\Lambda\| = \|\lambda_1\lambda_2\ldots\lambda_m\|^{\mathrm{T}}$ should be found, respectively, in the following pair of problems that is equivalent to problem (1.1), (1.2):

1. *Primal* linear program

maximize the function $F(x) = \sum_{j=1}^{n} \sigma_j \xi_j$ over $\{\xi_1, \xi_2, \ldots, \xi_n\}$

$$\text{subject to the constarints } \xi_j \geq 0 \quad \forall j = [1, n], \quad \text{where} \quad f_i(x) = -\beta_i + \sum_{j=1}^{n} \alpha_{ij}\xi_j \leq 0 \quad \forall i = [1, m]. \tag{2.1}$$

Any solution of problem (2.1) is denoted by x^*, and $F(x^*)$ is denoted by F^*.

2. *Dual* linear program

minimize the function $G(\Lambda) = \sum_{i=1}^{m} \beta_i \lambda_i$ over $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$

$$\text{subject to the constraints } \lambda_i \geq 0 \quad \forall i = [1, m], \text{ where} \quad g_j(\Lambda) = -\sigma_j + \sum_{i=1}^{m} \alpha_{ij}\lambda_i \geq 0 \quad \forall j = [1, n]. \tag{2.2}$$

Its solution will be denoted by Λ^*, and let $G^* = G(\Lambda^*)$.

The further presentation can be simplified if we first outline how the pair of problems (2.1), (2.2) can be solved using the following version of the smooth penalty function method [1].

Assume that this method uses the auxiliary functions

$$\begin{aligned} A_{\mathrm{P}}(\tau, x) &= F(x) - \sum_{i=1}^{m} P(\tau, f_i(x)) - \sum_{j=1}^{n} P(\tau, (-\xi_j)), \\ A_{\mathrm{D}}(\tau, \Lambda) &= G(\Lambda) + \sum_{j=1}^{n} P(\tau, -g_j(\Lambda)) + \sum_{i=1}^{m} P(\tau, (-\lambda_i)), \end{aligned} \tag{2.3}$$

where the function $P(\tau, s)$ determines the "penalty" for violating the constraint $s \le 0$ and satisfies the following conditions.

$$\begin{cases} \text{1. } \forall s \text{ and } \forall \tau > 0\text{: } P(\tau, s) \ge 0 \text{ and } \lim\limits_{\tau \to +0} P(\tau, s) \begin{cases} +\infty, & s > 0, \\ 0, & s < 0. \end{cases} \\ \text{2. The function } P(\tau, s) \text{ has continuous partial derivatives with respect to all its arguments} \\ \text{up to the second order, inclusive.} \\ \text{3. For all } \tau > 0 \text{ and } \forall s, \text{ it holds that } \dfrac{\partial P}{\partial s} > 0;\ \dfrac{\partial^2 P}{\partial s^2} > 0. \end{cases} \tag{2.4}$$

Let

$$\operatorname*{grad}_{x} A_{\mathrm{P}}(\tau, \tilde{x}(\tau)) = o \quad \text{and} \quad \operatorname*{grad}_{\Lambda} A_{\mathrm{D}}(\tau, \breve{\Lambda}(\tau)) = o. \tag{2.5}$$

Then, the quantities $F(\tilde{x}(\tau))$ and $G(\breve{\Lambda}(\tau))$, where the points $\tilde{x}(\tau)$ and $\breve{\Lambda}(\tau)$ are stationary for the auxiliary functions (2.3) $\forall \tau > 0$, can be used, due to the main property of the penalty function method, as approximations of $F(x^*)$ and $G(\Lambda^*)$. Moreover, the stationarity conditions for functions (2.5) satisfy the implicit function theorem (e.g., see Theorem 2 in [2, § 41]); therefore, the vector functions $\overline{x}(\tau)$ and $\breve{\Lambda}(\tau)$ can be considered as implicitly determined by Eqs. (2.5).

Finally, in the regular case (i.e., if the pair of problems (2.1), (2.2) has a unique solution), $\tilde{x}(\tau)$ can be used as an approximation of x^* and $\breve{\Lambda}(\tau)$ can be used as an approximation of Λ^*. As is shown, e.g., in [3], it holds that

$$\lambda_i^* = \lim_{\tau \to +0} \frac{\partial P}{\partial f_i}(\tau, f_i(\tilde{x}(\tau))) \quad \forall i = [1, m], \quad \xi_j^* = \lim_{\tau \to +0} \frac{\partial P}{\partial g_j}(\tau, -g_j(\breve{\Lambda}(\tau))) \quad \forall j = [1, n].$$

Now we describe the proposed approach using the reasoning presented in [4].

A property of the mutually dual problems (2.1) and (2.2) is that the components of the vector x^* are the Lagrange multipliers in problem (2.2), and the components of the vector Λ^* are the Lagrange multipliers in problem (2.1).However, at each fixed $\tau > 0$, we generally have

$$\frac{\partial P}{\partial f_i}(\tau, f_i(\tilde{x}(\tau))) \ne \breve{\lambda}_i(\tau)) \quad \forall i = [1, m] \quad \text{and} \quad \frac{\partial P}{\partial g_j}(\tau, -g_j(\breve{\Lambda}(\tau))) \ne \tilde{\xi}_j(\tau)) \quad \forall j = [1, n].$$

These relations become true only in the limit as $\tau \to +0$.

We may assume that there is a $\tau_0 > 0$ for which there exist vector functions $\overline{x}(\tau)$ and $\overline{\Lambda}(\tau)$ that $\forall \tau \in (0, \tau_0]$ are solutions to the system of equations

$$\begin{cases} \overline{\lambda}_i(\tau) = \dfrac{\partial P}{\partial s}(\tau, f_i(\overline{x}(\tau))) & \forall i = [1, m], \\ \overline{\xi}_j(\tau) = \dfrac{\partial P}{\partial s}(\tau, -g_j(\overline{\Lambda}(\tau))) & \forall j = [1, n], \end{cases} \tag{2.6}$$

for which (if problems (2.1) and (2.2) are consistent) it holds that $\lim\limits_{\tau \to +0} F(\overline{x}(\tau)) = F^*$ and $\lim\limits_{\tau \to +0} G(\overline{\Lambda}(\tau)) = G^*$, and, in the regular case (i.e., if x^* and Λ^* are unique), it also holds that

$$\lambda_i^* = \lim_{\tau \to +0} \frac{\partial P}{\partial f_i}(\tau, f_i(\overline{x}(\tau))) \quad \forall i = [1, m], \quad \xi_j^* = \lim_{\tau \to +0} \frac{\partial P}{\partial g_j}(\tau, -g_j(\overline{\Lambda}(\tau))) \quad \forall j = [1, n].$$

Table 1. Solutions to system (2.8) in Example 1 with the parameter $v = 6$

τ	$\bar{\xi}_1(\tau)$	$\bar{\xi}_2(\tau)$	$F(\bar{x}(\tau))$	$\bar{\lambda}_1(\tau)$	$\bar{\lambda}_2(\tau)$	$G(\bar{\Lambda}(\tau))$
10^{-1}	1.91387303	2.05644660	9.99708585	1.30690566	0.31409072	9.72597830
10^{-2}	1.99167722	2.00559101	10.0001275	1.33099033	0.33105995	9.97230168
10^{-3}	1.99917130	2.00055811	10.0000169	1.33310196	0.33310265	9.99722768
10^{-4}	1.99991717	2.00005580	10.0000017	1.33331023	0.33331023	9.99972274
10^{-5}	1.99999172	2.00000558	10.0000002	1.33333102	0.33333102	9.99997227
10^{-6}	1.99999917	2.00000056	10.0000000	1.33333310	0.33333102	9.99999723

Then, the vector functions $\bar{x}(\tau)$ and $\bar{\Lambda}(\tau)$, along with $\tilde{x}(\tau)$ and $\breve{\Lambda}(\tau)$, could be used as approximations of solutions to problems (2.1) and (2.2).

Note that system (2.6) can also be written as

$$\begin{aligned} f_i(\tau,\bar{x}(\tau)) &= Q(\tau,\bar{\lambda}_i(\tau)) \quad \forall i=[1,m], \\ -g_j(\tau,\bar{\Lambda}(\tau)) &= Q(\tau,\bar{\xi}_j(\tau)) \quad \forall j=[1,n] \end{aligned} \quad \text{or} \quad \begin{aligned} -\beta_i + \sum_{j=1}^{n}\alpha_{ij}\bar{\xi}_j &= Q(\tau,\bar{\lambda}_i) \quad \forall i=[1,m], \\ -\sigma_j + \sum_{i=1}^{m}\alpha_{ij}\bar{\lambda}_i &= -Q(\tau,\bar{\xi}_j) \quad \forall j=[1,n], \end{aligned} \tag{2.7}$$

where the function $Q(\tau,s) = \operatorname{inv}\left(\frac{\partial P}{\partial s}(\tau,s)\right)$ is the inverse of the function $\frac{\partial P}{\partial s}(\tau,s)$.

The probable validity of this assumption is confirmed by the following example.

Example 1. Solve the pair of problems with the parameter v, where

the primal problem is to maximize the function $F(x) = 2\xi_1 + 3\xi_2$ in E^2

subject to the constraints $\xi_1 \ge 0$, $\xi_2 \ge 0$ and $\xi_1 + 2\xi_2 \le v$, $2\xi_1 + \xi_2 \le 6$;

the dual problem is to minimize the function $G(\Lambda) = v\lambda_1 + 6\lambda_2$ in E^2

subject to the constraints $\lambda_1 \ge 0$, $\lambda_2 \ge 0$ and $\lambda_1 + 2\lambda_2 \ge 2$, $2\lambda_1 + \lambda_2 \ge 3$.

For $v = 6$, the solutions are $\xi_1^* = 2$, $\xi_2^* = 2$, $F^* = 10$ and $\lambda_1^* = \frac{4}{3}$, $\lambda_2^* = \frac{1}{3}$, $G^* = 10$.

In the case $P(\tau,s) = \tau\exp\left(\frac{s}{\tau}\right)$ and the corresponding $Q(\tau,s) = \operatorname{inv}\left(\exp\left(\frac{s}{\tau}\right)\right) = \tau\ln s$, system (2.7) has the form

$$\begin{aligned} -6 + \bar{\xi}_1 + 2\bar{\xi}_2 &= \tau\ln\bar{\lambda}_1, \\ -6 + 2\bar{\xi}_1 + \bar{\xi}_2 &= \tau\ln\bar{\lambda}_2, \\ -2 + \bar{\lambda}_1 + 2\bar{\lambda}_2 &= -\tau\ln\bar{\xi}_1, \\ -3 + 2\bar{\lambda}_1 + \bar{\lambda}_2 &= -\tau\ln\bar{\xi}_2; \end{aligned} \tag{2.8}$$

the solutions to this system for various values of the parameter τ are shown in Table 1.

In the next sections of this paper, we consider the conditions under which the methods based on the solution of systems similar to (2.7) are valid. Here we only note that the structure of this system implies that $Q(\tau,s)$ actually implements the feedback in the set of constraints for the primal and dual variables in problems (2.1) and (2.2). For this reason, we for brevity will use the term *feedback* when referring the functions of type $Q(\tau,s)$ and the methods for solving problem (1.1), (1.2) that use these functions.

3. PROOF OF VALIDITY OF THE METHOD OF FEEDBACKS FOR LINEAR PROGRAMS

Define the function $P(\tau,s)$ such that it satisfies conditions 1–3 in (2.4). Then, the continuously differentiable function $\frac{\partial P}{\partial s}$ monotonically increases in s $\forall s$ and has a continuously differentiable inverse function $Q(\tau, s)$, which, in turn, monotonically increases in s $\forall s > 0$. Furthermore, for this function we have $\lim_{s\to+0} Q(\tau, s) = -\infty$ $\forall\tau > 0$, $\lim_{s\to+\infty} Q(\tau, s) = +\infty$ $\forall\tau > 0$, and $\lim_{\tau\to+0} Q(\tau, s) = 0$ $\forall s > 0$.

Denote by $R(\tau, s)$ a nonnegative functions that has a unique zero and satisfies the equality

$$\frac{\partial R}{\partial s} = Q(\tau, s). \tag{3.1}$$

Note that under the assumptions made above, the function $R(\tau, s)$ exists and is unique.

Consider the auxiliary function (which will for brevity be called the U-function)

$$U(\tau, x, \Lambda) = \sum_{j=1}^{n}(\sigma_j\xi_j - R(\tau, \xi_j)) + \sum_{i=1}^{m}(\beta_i\lambda_i + R(\tau, \lambda_i)) - \sum_{j=1}^{n}\sum_{i=1}^{m}\alpha_{ij}\xi_j\lambda_i. \tag{3.2}$$

Note that, in this case, the solutions to system (2.7) (if they exist)—the vectors $\bar{x}(\tau)$ and $\bar{\Lambda}(\tau)$—are stationary points of the U-function with respect to the variables $\{x; \Lambda\}$.

Theorem 3.1. *The function $U(\tau, x, \Lambda)$ is strictly convex upwards with respect to x and strictly convex downwards with respect to Λ at every finite point with positive coordinates in the space $E^n \otimes E^m$ $\forall\tau > 0$.*

Proof. It follows from (3.2) and (2.7) that the elements of the Hessian matrix for $U(\tau, x, \Lambda)$ (as functions of x at fixed τ and Λ) are

$$\frac{\partial^2 U}{\partial\xi_j\partial\xi_k} = \frac{\partial Q}{\partial\xi_j}\delta_{jk} \quad \forall j, k = [1, n]$$

(where δ_{jk} is the Kronecker symbol). Then, its main minors have the form

$$\det\begin{Vmatrix} \frac{\partial^2 U}{\partial\xi_1^2} & \cdots & \frac{\partial^2 U}{\partial\xi_1\partial\xi_k} \\ \cdots & \cdots & \cdots \\ \frac{\partial^2 U}{\partial\xi_k\partial\xi_1} & \cdots & \frac{\partial^2 U}{\partial\xi_k^2} \end{Vmatrix} = \prod_{j=1}^{k}\left(-\frac{\partial Q}{\partial\xi_j}\right) \quad \forall k = [1, n].$$

The numbers $\frac{\partial Q}{\partial\xi_j}$ are positive; indeed, due to the relation between the derivatives of mutually inverse functions and Condition 3 in (2.4), we have

$$\frac{\partial Q}{\partial s}\left(\frac{\partial^2 P}{\partial s^2}\right)^{-1} > 0. \tag{3.3}$$

Therefore, all main minors of odd orders in the Hessian matrix under examination are negative, and the minors of even orders are positive. Consequently, due to the Sylvester criterion, the Hessian matrix of the function $U(\tau, x, \Lambda)$ is negative definite with respect to the components of x. Therefore, the function $U(\tau, x, \Lambda)$ is strictly convex upwards with respect to x.

Reasoning similarly, we prove that the main minors of the Hessian submatrix of $U(\tau, x, \Lambda)$ with respect to the components of Λ are $\prod_{i=1}^{k}\frac{\partial Q}{\partial\lambda_i}$ $\forall k = [1, m]$. Due to (3.3), they are positive. Consequently, due to the Sylvester criterion, the Hessian submatrix of the function $U(\tau, x, \Lambda)$ with respect to the components of Λ is positive definite, and the function $U(\tau, x, \Lambda)$ itself is strictly convex downwards for all fixed τ and x.

Now, we prove the consistency of system (2.7).

Theorem 3.2. *The system of equations* (2.7) *has a unique solution with positive components for every fixed $\tau > 0$ for any pair of problems* (2.1), (2.2).

Proof. 1. For each vector Λ with finite positive components, there exists a unique finite vector $\hat{x}(\Lambda)$ with positive components such that $\operatorname{grad}_x U(\tau, \hat{x}(\Lambda), \Lambda) = o$; due to (3.2), this equality can be written as

$$Q(\tau, \hat{\xi}_j(\Lambda)) - \sigma_j + \sum_{i=1}^{m}\alpha_{ij}\lambda_i = 0 \quad \forall j = [1, n]. \tag{3.4}$$

Indeed, $U(\tau, x, \Lambda)$ is strictly convex upwards with respect to x and conditions (3.4) are equivalent to the equalities

$$\hat{\xi}_j(\Lambda) = \frac{\partial P}{\partial s}(\tau, -g_j(\Lambda)) \quad \forall j = [1, n], \quad \forall \Lambda > 0,$$

in which the right-hand sides exist, are positive, and determined uniquely. This, in turn, implies that $\hat{x}(\Lambda) = \operatorname*{argmax}_x U(\tau, x, \Lambda)$.

Reasoning similarly, we conclude that, for every finite vector x with positive components, there exists a finite vector $\hat{\Lambda}(x)$ with positive components such that

$$Q(\tau, \hat{\lambda}_i(x)) + \beta_i - \sum_{j=1}^{n} \alpha_{ij}\xi_j = 0 \quad \forall i = [1, m] \tag{3.5}$$

and that $\hat{\Lambda}(x) = \operatorname*{argmin}_\Lambda U(\tau, x, \Lambda)$ due to the strict convexity of $U(\tau, x, \Lambda)$ downwards with respect to Λ.

2. Now we find the minimum of $U(\tau, \hat{x}(\Lambda), \Lambda)$ with respect to Λ. Formula (3.2) implies

$$U(\tau, \hat{x}(\Lambda), \Lambda) = \sum_{j=1}^{n} (\sigma_j \hat{\xi}_j(\Lambda) - R(\tau, \hat{\xi}_j(\Lambda))) + \sum_{i=1}^{m} (\beta_i \lambda_i + R(\tau, \lambda_i)) - \sum_{j=1}^{n}\sum_{i=1}^{m} \alpha_{ij}\hat{\xi}_j(\Lambda)\lambda_i.$$

Due to the strict convexity of this function downwards, a sufficient condition for its minimum is

$$\frac{\partial U}{\partial \lambda_q}(\tau, \hat{x}(\Lambda), \Lambda) = \sum_{j=1}^{n}\left[\sigma_j \frac{\partial \hat{\xi}_j(\Lambda)}{\partial \lambda_q} - \frac{\partial R}{\partial s}(\tau, \hat{\xi}_j(\Lambda))\frac{\partial \hat{\xi}_j(\Lambda)}{\partial \lambda_q}\right]$$
$$+ \beta_q + \frac{\partial R}{\partial s}(\tau, \lambda_q) - \sum_{j=1}^{n} \alpha_{qj}\hat{\xi}_j(\Lambda) - \sum_{j=1}^{n}\sum_{i=1}^{m} \alpha_{qj}\lambda_i \frac{\partial \hat{\xi}_j(\Lambda)}{\partial \lambda_q} = 0 \quad \forall q = [1, m]$$

or, after regrouping the terms,

$$\sum_{j=1}^{n} \frac{\partial \hat{\xi}_j(\Lambda)}{\partial \lambda_q}\left[\sigma_j - Q(\tau, \hat{\xi}_j(\Lambda)) - \sum_{i=1}^{m} \alpha_{ij}\lambda_i\right] + \beta_q + Q(\tau, \lambda_q) - \sum_{j=1}^{n} \alpha_{qj}\hat{\xi}_j(\Lambda) = 0 \quad \forall q = [1, m], \tag{3.6}$$

because $\forall j = [1, n]$ and $\forall q = [1, m]$ we have

$$\frac{\partial R}{\partial s}(\tau, \hat{\xi}_j(\Lambda)) = Q(\tau, \hat{\xi}_j(\Lambda)) \quad \text{and} \quad \frac{\partial R}{\partial s}(\tau, \lambda_q) = Q(\tau, \lambda_q).$$

Hence, due to (3.4), equality (3.6), which is a sufficient condition for the minimum of the function $U(\tau, \hat{x}(\Lambda), \Lambda)$ with respect to Λ, can be simplified as

$$\beta_q + Q(\tau, \hat{\hat{\lambda}}_q) - \sum_{j=1}^{n} \alpha_{qj}\hat{\xi}_j(\hat{\hat{\Lambda}}) = 0 \quad \forall q = [1, m], \tag{3.7}$$

where $\hat{\hat{\Lambda}} = \operatorname*{argmin}_\Lambda U(\tau, \hat{x}(\Lambda), \Lambda)$ is the vector with nonnegative components satisfying the equality $\min_\Lambda \max_x U(\tau, x, \Lambda) = U(\tau, \hat{x}(\Lambda), \hat{\hat{\Lambda}})$ and such that $\|\hat{\hat{\Lambda}}\| = \|\hat{\hat{\lambda}}_1 \hat{\hat{\lambda}}_2 \ldots \hat{\hat{\lambda}}_m\|^{\mathrm{T}}$ exists and is unique due to the strict convexity of $U(\tau, x, \Lambda)$ downwards with respect to Λ.

Let us now find the minimum of $U(\tau, \hat{x}(\Lambda), \Lambda)$ with respect to Λ. We have

$$U(\tau, \hat{x}(\hat{\hat{\Lambda}}), \hat{\hat{\Lambda}}) = \sum_{j=1}^{n} (\sigma_j \hat{\xi}_j(\hat{\hat{\Lambda}}) - R(\tau, \hat{\xi}_j(\hat{\hat{\Lambda}}))) + \sum_{i=1}^{m} (\beta_i \hat{\hat{\lambda}}_i + R(\tau, \hat{\hat{\lambda}}_i)) - \sum_{j=1}^{n}\sum_{i=1}^{m} \alpha_{ij}\hat{\xi}_j(\hat{\hat{\Lambda}})\hat{\hat{\lambda}}_i.$$

By regrouping the terms, we obtain

$$U(\tau, \hat{x}(\hat{\hat{\Lambda}}), \hat{\hat{\Lambda}}) = \sum_{i=1}^{m} (\beta_i \hat{\hat{\lambda}}_i + R(\tau, \hat{\hat{\lambda}}_i)) + \sum_{j=1}^{n}\left(\sigma_j - \sum_{i=1}^{m} \alpha_{ij}\hat{\hat{\lambda}}_i\right)\hat{\xi}_j(\hat{\hat{\Lambda}}) - \sum_{j=1}^{n} R(\tau, \hat{\xi}_j(\hat{\hat{\Lambda}})).$$

Finally, due to (3.4), we obtain

$$U(\tau, \hat{x}(\hat{\hat{\Lambda}}), \hat{\hat{\Lambda}}) \min_{\Lambda} \max_{x} U(\tau, x, \Lambda) \sum_{i=1}^{m} (\beta_i \hat{\hat{\lambda}}_i + R(\tau, \hat{\hat{\lambda}}_i)) + \sum_{j=1}^{n} (\hat{\xi}_j(\hat{\hat{\Lambda}}) Q(\tau, \hat{\xi}_j(\hat{\hat{\Lambda}})) - R(\tau, \hat{\xi}_j(\hat{\hat{\Lambda}})). \tag{3.8}$$

3. Now, using the reasoning similar to that used in item 2, we find $\max_{x} \min_{\Lambda} U(\tau, x, \Lambda)$.

Since $U(\tau, x, \Lambda)$ is a strictly convex downward function with respect to the components of the vector Λ, a sufficient condition for its minimum with respect to Λ is

$$-Q(\tau, \hat{\lambda}_i(x)) = \beta_i - \sum_{j=1}^{n} \alpha_{ij} \xi_j \quad \forall i = [1, m]. \tag{3.9}$$

The minimum of $U(\tau, x, \Lambda)$ with respect to Λ at fixed τ and x is

$$U(\tau, x, \hat{\Lambda}(x)) = \sum_{j=1}^{n} (\sigma_j \xi_j - R(\tau, \xi_j))) + \sum_{i=1}^{m} (\beta_i \hat{\lambda}_i(x) + R(\tau, \hat{\lambda}_i(x))) - \sum_{j=1}^{n} \sum_{i=1}^{m} \alpha_{ij} \xi_j \hat{\lambda}_i(x).$$

Due to the upwards convexity of this function with respect to x, a sufficient condition for its maximum with respect to x (after regrouping the terms) is

$$\sum_{i=1}^{m} \frac{\partial \hat{\lambda}_i(x)}{\partial \xi_k} \left[\beta_i + Q(\tau, \hat{\lambda}_i(x)) - \sum_{j=1}^{n} \alpha_{ij} \xi_j \right] + \sigma_k - Q(\tau, \xi_k) - \sum_{i=1}^{m} \alpha_{ik} \hat{\lambda}_i(x) = 0 \quad \forall k = [1, n]. \tag{3.10}$$

The expressions in brackets in (3.10) vanish due to (3.9); therefore, the sufficient condition for the maximum of $U(\tau, x, \hat{\Lambda}(x))$ with respect to x takes the form

$$\sigma_k - Q(\tau, \hat{\hat{\xi}}_k) - \sum_{i=1}^{m} \alpha_{ik} \hat{\lambda}_i(\hat{\hat{\xi}}_k) = 0 \quad \forall k = [1, n], \tag{3.11}$$

where the vector $\hat{\hat{x}} = \operatorname{argmax}_{x} U(\tau, x, \hat{\Lambda}(x))$ with nonnegative components for which it holds that $\max_{x} \min_{\Lambda} U(\tau, x, \Lambda) = U(\tau, \hat{\hat{x}}, \hat{\Lambda}(x))$ exists and is unique due to the strict convexity upwards of the function $U(\tau, x, \Lambda)$ with respect to x.

For the maximum of $U(\tau, x, \hat{\Lambda}(x))$ with respect to x (after regrouping the terms and taking into account (3.11)), we finally have

$$U(\tau, \hat{\hat{x}}, \hat{\Lambda}(\hat{\hat{x}})) \max_{x} \min_{\Lambda} U(\tau, x, \Lambda) \sum_{i=1}^{m} (\beta_i \hat{\lambda}_i(\hat{\hat{x}}) + R(\tau, \hat{\lambda}_i(\hat{\hat{x}}))) + \sum_{j=1}^{n} (\hat{\hat{\xi}}_j Q(\tau, \hat{\hat{\xi}}_j) - R(\tau, \hat{\hat{\xi}}_j)). \tag{3.12}$$

4. Note that conditions (3.7) and (3.9) imply that $\hat{\hat{\Lambda}} = \hat{\Lambda}(\hat{\hat{x}})$, and conditions (3.4) and (3.11) similarly imply that $\hat{\hat{x}} = \hat{x}(\hat{\hat{\Lambda}})$ because (3.7) is a sufficient condition for the maximum of $U(\tau, x, \hat{\Lambda}(x))$ with respect to x and (3.11) is a sufficient condition for the minimum of $U(\tau, \hat{x}(\Lambda), \Lambda)$ with respect to Λ.

We illustrate these assertions by deriving, by way of example, formula $\hat{\hat{x}} = \hat{x}(\hat{\hat{\Lambda}})$. To this end, we rewrite (3.4) and (3.11) by replacing in the latter one the index k by j:

$$\begin{aligned} -Q(\tau, \hat{\xi}_j(\Lambda)) &= -\sigma_j + \sum_{i=1}^{m} \alpha_{ij} \lambda_i \quad \forall j = [1, n], \\ -Q(\tau, \hat{\hat{\xi}}_j) &= -\sigma_j + \sum_{i=1}^{m} \alpha_{ij} \hat{\lambda}_i(\hat{\hat{\xi}}_j) \quad \forall j = [1, n]. \end{aligned} \tag{3.13}$$

The first group of equalities (3.13) holds for all positive $\lambda_i \forall i = [1, m]$, including $\hat{\lambda}_i(\hat{\hat{\xi}}_j) \forall i = [1, m]$. It is clear that, in this case, the right-hand sides of these equalities are identical. Therefore, the left-hand sides are also identical; i.e.

$$Q(\tau, \hat{\xi}_j(\hat{\hat{\Lambda}})) = Q(\tau, \hat{\hat{\xi}}_j) \quad \forall j = [1, n].$$

This equality implies that, due to the monotonicity of the function $Q(\tau, s)$ in s, it holds that $\hat{\xi}_j(\hat{\hat{\Lambda}}) = \hat{\hat{\xi}}_j$ $\forall j = [1, n]$, i.e., $\hat{\hat{x}} = \hat{x}(\hat{\hat{\Lambda}})$.

By comparing formulas (3.8) and (3.12), we conclude that

$$\min_{\Lambda} \max_{x} U(\tau, x, \Lambda) = \max_{x} \min_{\Lambda} U(\tau, x, \Lambda).$$

Therefore, the function $U(\tau, x, \Lambda)$ has a saddle point, which is a stationary point of $U(\tau, x, \Lambda)$, because this function is continuously differentiable. The stationarity conditions can be written in the form

$$\operatorname*{grad}_{x} U = o,$$
$$\operatorname*{grad}_{\Lambda} U = o,$$

which is equivalent to system (2.7). This completes the proof.

Below (if no otherwise specified), we assume not only that the function $P(\tau, s)$ satisfies conditions (2.4) but also that $\frac{\partial P}{\partial s}(\tau, s)$ is a function of the single argument $u = \frac{s}{\tau}$, i.e.,

$$\frac{\partial P}{\partial s}(\tau, s) = \Phi\left(\frac{s}{\tau}\right), \tag{3.14}$$

where $\Phi(u)$ is a function defined $\forall u$ that guarantees the fulfillment of conditions (2.4).

Formally, condition (3.14) is an additional constraint of generality in the choice of the function $P(\tau, s)$, which does not play an important role because $P(\tau, s)$ is an auxiliary function; however, it not only simplifies the proof of validity of the method of feedbacks but also improves its convergence as $\tau \to +0$ as was shown in [5].

Now, consider the properties of the function $R(\tau, s)$. It is clear that this function is defined for all positive τ and s, is nonnegative in its domain, has a unique zero, and is twice continuously differentiable and strictly convex downwards with respect to s for $s > 0$. An important property of $R(\tau, s)$ is given by the following theorem.

Theorem 3.3. *For every fixed* $s \in (0, +\infty)$, *it holds that* $\lim_{\tau \to +0} R(\tau, s) = 0$.

Proof. Due to (3.14), the feedback function $Q(\tau, s)$ is determined by the equation $\Phi\left(\frac{Q}{\tau}\right) = s$, and it has the form $Q(\tau, s) = \tau\Psi(s)$, where $\Psi(u)$ is the inverse function of $\Phi(u)$ defined on the interval $(0, +\infty)$.

Then, for any finite positive s,

$$R(\tau, s) = \int_a^s Q(\tau, u)du = \tau\int_a^s \Psi(u)du, \quad \text{where} \quad Q(\tau, a) = 0,$$

implies the validity of the theorem assertion.

Another useful property of the U-function is given by the following theorem

Theorem 3.4. *If* $L(x, \Lambda)$ *is the Lagrangian function of the pair of problems* (2.1), (2.2)*, then,* $\forall x$ *and* $\forall \Lambda$ *in the domain of* $L(x, \Lambda)$*, it holds that*

$$\lim_{\tau \to +0} U(\tau, x, \Lambda) = L(x, \Lambda). \tag{3.15}$$

Proof. The Lagrangian function of the pair of problems (2.1), (2.2) is

$$L(x, \Lambda) = \sum_{j=1}^{n} \sigma_j \xi_j - \sum_{i=1}^{m} \lambda_i \left(-\beta_i + \sum_{j=1}^{n} \alpha_{ij}\xi_j\right) \quad \text{or} \quad L(x, \Lambda) = \sum_{j=1}^{n} \sigma_j \xi_j + \sum_{i=1}^{m} \beta_i \lambda_i - \sum_{i=1}^{m}\sum_{j=1}^{n} \alpha_{ij}\xi_j\lambda_i.$$

Therefore, the U-function can be written as

$$U(\tau, x, \Lambda) = L(x, \Lambda) - \sum_{j=1}^{n} R(\tau, \xi_j) + \sum_{i=1}^{m} R(\tau, \lambda_i). \tag{3.16}$$

Applying the assertion of Theorem 3.3 to formula (3.16), we obtain the assertion of Theorem 3.4.

Definition. The set of points $\{\bar{x}(\tau);\bar{\Lambda}(\tau)\}$ $\forall\tau>0$ in the space $E^n\otimes E^m$ is called the saddle trajectory of the U-function of the pair of problems (2.1), (2.2).

Theorem 3.2 implies that the vector functions $\bar{x}(\tau)$, $\bar{\Lambda}(\tau)$ and the scalar function $\bar{U}(\tau)=U(\tau,\bar{x}(\tau),\bar{\Lambda}(\tau))$ are defined on each saddle trajectory. Let us consider their properties.

Theorem 3.5. *For the proper problems* (2.1) *and* (2.2)*, i.e., for the problems that have bounded optimal values of the objective functions, the vector functions* $\bar{x}(\tau)$ *and* $\bar{\Lambda}(\tau)$ *have bounded components set* $\forall\tau>0$.

Proof. 1. Based on (2.6) and using (2.1) and (2.2), the stationarity conditions for the auxiliary U-function (3.2) can be written as

$$\begin{aligned}\bar{\lambda}_i(\tau)&=\frac{\partial P}{\partial s}\left(\tau,-\beta_i+\sum_{j=1}^{n}\alpha_{ij}\frac{\partial P}{\partial s}\left(\tau,\sigma_j-\sum_{q=1}^{m}\alpha_{qj}\bar{\lambda}_q(\tau)\right)\right)\quad\forall i=[1,m],\\ \bar{\xi}_j(\tau)&=\frac{\partial P}{\partial s}\left(\tau,\sigma_j-\sum_{i=1}^{m}\alpha_{ij}\frac{\partial P}{\partial s}\left(\tau,-\beta_i+\sum_{p=1}^{n}\alpha_{ip}\bar{\xi}_p(\tau)\right)\right)\quad\forall j=[1,n];\end{aligned}\tag{3.17}$$

i.e., system (2.7) splits into two independent subsystems; in the first subsystem, the unknowns are the components of the vector function $\bar{\Lambda}(\tau)$ and in the second subsystem, the unknowns are the components of $\bar{x}(\tau)$.

Let us examine in more detail the second subsystem and introduce, for convenience, the scalar functions

$$W_j(\tau,\bar{x}(\tau))=\sigma_j-\sum_{i=1}^{m}\alpha_{ij}\frac{\partial P}{\partial s}(\tau,V_i(\bar{x}(\tau)))\quad\forall j=[1,n]\quad\text{and}\quad V_i(\bar{x}(\tau))-\beta_i+\sum_{p=1}^{n}\alpha_{ip}\bar{\xi}_p(\tau)\quad\forall i=[1,m].$$

In this case, the second group of equations in (3.17) takes the form

$$\bar{\xi}_j(\tau)=\frac{\partial P}{\partial s}(\tau,W_j(\tau,\bar{x}(\tau)))\quad\forall j=[1,n],\tag{3.18}$$

and the partial derivative of the function $\frac{\partial P}{\partial s}(\tau,W_j(\tau,x))$ with respect to ξ_k is

$$\begin{aligned}\frac{\partial}{\partial\xi_k}\frac{\partial P}{\partial s}(\tau,W_j(\tau,x))&=\frac{\partial^2P}{\partial s^2}(\tau,W_j(\tau,x))\frac{\partial W_j}{\partial\xi_k}=\frac{\partial^2P}{\partial s^2}(\tau,W_j(\tau,x))\frac{\partial}{\partial\xi_k}\left(\sigma_j-\sum_{i=1}^{m}\alpha_{ik}\frac{\partial P}{\partial s}(\tau,V_i(\tau,x))\right)\\&=-\frac{\partial^2P}{\partial s^2}(\tau,W_j(\tau,x))\left(\sum_{i=1}^{m}\alpha_{ik}\frac{\partial^2P}{\partial s^2}(\tau,x)\frac{\partial V_i}{\partial\xi_k}\right)=-\frac{\partial^2P}{\partial s^2}(\tau,W_j(\tau,x))\left(\sum_{i=1}^{m}\alpha_{ik}\frac{\partial^2P}{\partial s^2}(\tau,x)\alpha_{ik}\right).\end{aligned}$$

Hence, taking into account that the second derivative $\frac{\partial^2P}{\partial s^2}(\tau,s)$ is defined and positive $\forall s$ due to (2.4), we obtain

$$\frac{\partial}{\partial\xi_k}\frac{\partial P}{\partial s}(\tau,W_j(\tau,x))=-\frac{\partial^2P}{\partial s^2}(\tau,W_j(\tau,x))\left(\sum_{i=1}^{m}\alpha_{ik}^2\frac{\partial^2P}{\partial s^2}(\tau,x)\right)\le 0;$$

moreover, for $\sum_{i=1}^{m}|\alpha_{ik}|>0$, this inequality is strict.

Thus, the right-hand side of the jth equation in (3.18) is a monotonically decreasing function of the kth component of the vector $\bar{x}(\tau)$.

2. Now consider the jth equation in (3.18) in which all unknowns, except for $\bar{\xi}_j$, are fixed. Let it have the form $\bar{\xi}_j=Z(\bar{\xi}_j)$. Note that the domain of the function $Z(\bar{\xi}_j)$ and the set of its values is the set of positive numbers. Item 1 of this proof implies that the continuous function $Z(\bar{\xi}_j)$ monotonically decreases on its entire domain. Therefore, it has a monotonically decreasing inverse function $T(\bar{\xi}_j)$.

Let $\bar{\xi}_j^*$ be the solution of the equation under consideration, i.e., $\bar{\xi}_j^*=Z(\bar{\xi}_j^*)$. It is clear that $T(\bar{\xi}_j^*)=\bar{\xi}_j^*$. If $\bar{\xi}_j^*$ is bounded from above, then $\exists+\infty>D_j>0:\bar{\xi}_j^*\le D_j$, and, therefore, $\bar{\xi}_j^*=T(\bar{\xi}_j^*)=Z(\bar{\xi}_j^*)\le D_j$. Then, due to the monotone decrease of the function $T(\bar{\xi}_j)$, we obtain $T(Z(\bar{\xi}_j^*))\ge T(D_j)$. In other words, $\exists d_j>0:d_j\le\bar{\xi}_j^*$, where $d_j=T(D_j)\le D_j$. Therefore, $\bar{\xi}_j^*$ is also bounded from below by a strictly positive number.

Using similar reasoning, we can show that the boundedness of the component $\overline{\xi}_j^*$ from below implies its boundedness from above.

3. Let us now prove that in the proper case, the solutions to system (3.17) are bounded $\forall\tau > 0$, and in the improper case they have unbounded components.

Indeed, if problems (2.1) and (2.2) are solvable and their solutions are finite, then the systems

$$\begin{matrix} \sigma_j - \sum\limits_{i=1}^{m} \alpha_{ij}\lambda_i \le 0 \quad \forall j = [1,n], \\ \lambda_i \ge 0 \quad \forall i = [1,m] \end{matrix} \quad \text{and} \quad \begin{matrix} -\beta_i + \sum\limits_{j=1}^{n} \alpha_{ij}\xi_j \le 0 \quad \forall i = [1,m], \\ \xi_j \ge 0 \quad \forall j = [1,n] \end{matrix} \tag{3.19}$$

are consistent.

Consider the second of these systems. In this case, for each $j = [1,n]$, there exists a vector $x_{(j)}$ such that $W_j(\tau, x_{(j)}) = 0$. Note that each left-hand side in Eqs. (3.18), which has a finite value for each fixed $\tau > 0$, due to Theorem 3.2, can be greater or less than $\Phi(0) = \dfrac{\partial P}{\partial s}(\tau, 0)$. Therefore, due to bounds in item 2, it is bounded both from below and from above by one of the three positive numbers $T(\Phi(0)), \Phi(0)$, or $D(\Phi(0))$, where each of them is independent of τ. by assumption (3.14). Therefore, all left-hand sides in (3.18) are uniformly bounded on the set $\tau \in (0, +\infty)$.

For the first system in (3.19), the reasoning is similar.

In the improper case, at least one of systems (3.19) is inconsistent. Assume that this is the second system. Then, for at least one index i, we have $f_i(x) > 0$ for all x with nonnegative components, including the solution $\overline{x}(\tau)$ to system (3.18), which exists due to Theorem 3.2.

Then, due to

$$\overline{\lambda}_i(\tau) = \frac{\partial P}{\partial s}(\tau, f_i(\overline{x}(\tau))) = \Phi\left(\frac{f_i(\overline{x}(\tau))}{\tau}\right),$$

it holds that

$$\lim_{\tau \to +0} \Phi\left(\frac{f_i(\overline{x}(\tau))}{\tau}\right) = +\infty,$$

which implies the existence of unbounded components in solutions to system (3.17) for the improper pair of problems (2.1), (2.2).

If the first system in (3.19) is improper, the reasoning is similar.

Theorem 3.6. *For the proper problems* (2.1) *and* (2.2), *the vector functions* $\overline{x}(\tau)$ *and* $\overline{\Lambda}(\tau)$ *are continuously differentiable* $\forall\tau > 0$.

Proof. 1. Let us introduce the notation

$$\kappa_j = \frac{\partial Q}{\partial \xi_j} \quad \forall j = [1,n] \quad \text{and} \quad \mu_i = \frac{\partial Q}{\partial \lambda_i} \quad \forall i = [1,m].$$

The Jacobian matrix of system (2.7), which coincides with the Hessian matrix of the auxiliary function (3.2), has in this case the form

$$\|H\| = \left\|\begin{array}{cccc|cccc} -\kappa_1 & 0 & \dots & 0 & -\alpha_{11} & -\alpha_{21} & \dots & -\alpha_{m1} \\ 0 & -\kappa_2 & \dots & 0 & -\alpha_{12} & -\alpha_{22} & \dots & -\alpha_{m2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -\kappa_n & -\alpha_{1n} & -\alpha_{2n} & \dots & -\alpha_{mn} \\ \hline -\alpha_{11} & -\alpha_{12} & \dots & -\alpha_{1n} & \mu_1 & 0 & \dots & 0 \\ -\alpha_{21} & -\alpha_{22} & \dots & -\alpha_{2n} & \mu_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -\alpha_{m1} & -\alpha_{m2} & \dots & -\alpha_{mn} & 0 & 0 & \dots & \mu_m \end{array}\right\|$$

or, in block notation, $\|H\| = \left\|\begin{matrix} -\|X\| & -\|A\|^{\mathrm{T}} \\ -\|A\| & \|Y\| \end{matrix}\right\|$.

The elements of the matrix $\|A\|$ are the numbers α_{ij} $\forall i = [1, m]$, $j = [1, n]$, and the elements of the diagonal matrices $\|X\|$ and $\|Y\|$ are $\delta_{ij}\kappa_j$ $\forall i, j = [1, n]$ and $\delta_{ij}\mu_i$ $\forall i, j = [1, m]$, respectively.

Let us find out the sign of $\det\|H\|$. By a property of the Schur complement (e.g. see [6, Chapter 4, § 6, Subsection 5]) and due to the rules of matrix manipulation, we have

$$\det\|H\| = \det\|X\| \cdot \det(\|Y\| + \|A\|\|X\|^{-1}\|A\|^{\mathrm{T}}).$$

The numbers κ_j $\forall j = [1, n]$ and μ_i $\forall i = [1, m]$ are positive because the function $Q(\tau, s)$is strictly monotonically increasing. Therefore, it is clear that $\det\|X\|^{-1} > 0$.

Let us now estimate the second factor $\det(\|Y\| + \|A\|\|X\|^{-1}\|A\|^{\mathrm{T}})$. Note that the matrix $\|Y\|$ can be considered as the matrix of a positive definite quadratic form in the space E^m, and the matrix $\|A\|\|X\|^{-1}\|A\|^{\mathrm{T}}$ (for any rank of the matrix $\|A\|$) specifies either a positive definite or positive semidefinite quadratic form in the same space due to a corollary to the Cauchy–Binet theorem (e.g. see [6, Chapter 4, § 5, Subsection 6]).

It is clear that in this case the matrix $(\|Y\| + \|A\|\|X\|^{-1}\|A\|^{\mathrm{T}})$ also specifies a positive definite quadratic form in E^m and has a positive determinant (due to the Sylvester criterion). Therefore, we finally obtain $\det\|H\| > 0$.

2. The nonsingularity of the Jacobian matrix for the system of equations (2.7) together with the assumptions on the smoothness of the function $Q(\tau, s)$ allows us to apply the implicit function theorem [2] to this system. This implies the continuity of the vector functions $\overline{x}(\tau)$ and $\overline{\Lambda}(\tau)$.

The components of the vector functions $\dfrac{d\overline{x}}{d\tau}$ and $\dfrac{d\overline{\Lambda}}{d\tau}$ are determined by the following system of linear equations $\forall \tau > 0$:

$$\left\|\begin{matrix} -\|X\| & -\|A\|^{\mathrm{T}} \\ -\|A\| & \|Y\| \end{matrix}\right\| \left\|\begin{matrix} \left\|\dfrac{d\overline{x}}{d\tau}\right\| \\ \left\|\dfrac{d\overline{\Lambda}}{d\tau}\right\| \end{matrix}\right\| = -\left\|\begin{matrix} \left\|\dfrac{\partial^2 U}{\partial x \partial \tau}\right\| \\ \left\|\dfrac{\partial^2 U}{\partial \Lambda \partial \tau}\right\| \end{matrix}\right\|, \tag{3.20}$$

where

$$\left\|\frac{d\overline{x}}{d\tau}\right\|_j = \frac{d\overline{\xi}_j}{d\tau} \quad \forall j = [1, n], \quad \left\|\frac{d\overline{\Lambda}}{d\tau}\right\|_i = \frac{d\overline{\lambda}_i}{d\tau} \quad \forall i = [1, m], \quad \left\|\frac{\partial^2 U}{\partial x \partial \tau}\right\|_j = -\frac{\partial Q}{\partial \tau}(\tau, \overline{\xi}_j(\tau)) \quad \forall j = [1, n]$$

$$\text{and} \quad \left\|\frac{\partial^2 U}{\partial \Lambda \partial \tau}\right\|_i = \frac{\partial Q}{\partial \tau}(\tau, \overline{\lambda}_i(\tau)) \quad \forall i = [1, m].$$

The last two functions are bounded and continuous due to Theorem 3.5. Therefore, (3.20) implies the continuous differentiability of the vector functions $\overline{x}(\tau)$ and $\overline{\Lambda}(\tau)$ on the set $\tau > 0$.

Corollary 3.1. *For the proper pair of problems* (2.1), (2.2)*, there exist finite limits* $\lim\limits_{\tau\to+0} \overline{x}(\tau)$, $\lim\limits_{\tau\to+0} \overline{\Lambda}(\tau)$*, and* $\lim\limits_{\tau\to+0} U(\tau, \overline{x}(\tau), \overline{\Lambda}(\tau))$ *on the saddle trajectory.*

Proof. The continuous differentiability and boundedness of the vector functions $\overline{x}(\tau)$ and $\overline{\Lambda}(\tau)$ $\forall \tau > 0$ in the proper case imply their uniform continuity on this set. Therefore, the limits mentioned in the corollary exist and are finite.

The properties of these limits are given by the following theorem.

Theorem 3.7. *For the proper pair of problems* (2.1), (2.2)*, the following equalities hold on the saddle trajectory:*

$$\lim_{\tau\to+0} \overline{\lambda}_i(\tau) f_i(\overline{x}(\tau)) = 0 \quad \forall i = [1, m], \quad \lim_{\tau\to+0} \hat{\xi}_j(\tau) g_j(\overline{\Lambda}(\tau)) = 0 \quad \forall j = [1, n],$$
$$\lim_{\tau\to+0} (F(\overline{x}(\tau)) - G(\overline{\Lambda}(\tau))) = 0.$$

Proof. 1. First, note that for finite $s > 0$, we have due to (3.14) the equality

$$\lim_{\tau\to+0} Q(\tau, s) = \lim_{\tau\to+0} \tau\Psi(s) = 0.$$

By multiplying both sides of (2.7) by $\overline{\lambda}_i$ $\forall i = [1, m]$ and $\overline{\xi}_j$ $\forall j = [1, n]$, respectively, we obtain

$$\overline{\lambda}_i(\tau) f_i(\overline{x}(\tau)) = \lambda_i(\tau) Q(\tau, \lambda_i(\tau)) \quad \forall i = [1, m],$$
$$\overline{\xi}_j(\tau) g_j(\overline{\Lambda}(\tau)) = \xi_j(\tau) Q(\tau, \xi_j(\tau)) \quad \forall j = [1, n].$$

Now, due to the remark made above, we obtain the first two assertions of the theorem.

2. The stationarity conditions of the U-function imply that

$$\sigma_j = Q(\tau, \overline{\xi}_j(\tau)) + \sum_{i=1}^{m} \alpha_{ij} \overline{\lambda}_i(\tau) \quad \forall j = [1, n];$$

therefore, we obtain

$$\sum_{j=1}^{n} \sigma_j \overline{\xi}_j(\tau) = \sum_{j=1}^{n} \overline{\xi}_j(\tau) Q(\tau, \overline{\xi}_j(\tau)) + \sum_{j=1}^{n}\sum_{i=1}^{m} \alpha_{ij} \overline{\lambda}_i(\tau) \overline{\xi}_j(\tau).$$

Similarly, we find that

$$\sum_{i=1}^{m} \beta_i \overline{\lambda}_i(\tau) = -\sum_{i=1}^{m} \overline{\lambda}_i(\tau) Q(\tau, \overline{\lambda}_i(\tau)) + \sum_{i=1}^{m}\sum_{j=1}^{n} \alpha_{ij} \overline{\xi}_j(\tau) \overline{\lambda}_i(\tau).$$

The term-by-term subtraction of the last two equalities gives

$$F(\overline{\xi}_j(\tau)) - G(\overline{\lambda}_i(\tau)) = \sum_{j=1}^{n} \overline{\xi}_j(\tau) Q(\tau, \overline{\xi}_j(\tau)) + \sum_{i=1}^{m} \overline{\lambda}_i(\tau) Q(\tau, \overline{\lambda}_i(\tau)) \quad \forall \tau > 0.$$

Therefore, due to item 1 of this proof, we have on the saddle trajectory the equality

$$\lim_{\tau\to+0} (F(\overline{x}(\tau)) - G(\overline{\Lambda}(\tau))) = 0.$$

Theorem 3.8. *On the saddle trajectories of the proper problems* (2.1) *and* (2.2),

$$\lim_{\tau\to+0} U(\tau, \overline{x}(\tau), \overline{\Lambda}(\tau)) = F(x^*) = G(\Lambda^*), \tag{3.21}$$

and in the case of uniqueness (*regularity*) *of the pair of problems* (2.1), (2.2), *it also holds that*

$$\lim_{\tau\to+0} \overline{x}(\tau) = x^* \quad \textit{and} \quad \lim_{\tau\to+0} \overline{\Lambda}(\tau) = \Lambda^*. \tag{3.22}$$

Proof. Due to (3.16), we have on the saddle trajectory

$$U(\tau, \overline{x}(\tau), \overline{\Lambda}(\tau)) = L(\overline{x}(\tau), \overline{\Lambda}(\tau)) - \sum_{j=1}^{n} R(\tau, \overline{\xi}_j(\tau)) + \sum_{i=1}^{m} R(\tau, \overline{\lambda}_i(\tau)) \quad \forall \tau > 0;$$

therefore, $\forall \tau > 0$ we have the bounds

$$U(\tau, \overline{x}(\tau), \overline{\Lambda}(\tau)) \le L(\overline{x}(\tau), \overline{\Lambda}(\tau)) + \sum_{i=1}^{m} R(\tau, \overline{\lambda}_i(\tau)) \le G^* + m \max_{i=[1,m]} R(\tau, \overline{\lambda}_i(\tau)) \tag{3.23}$$

and

$$U(\tau, \overline{x}(\tau), \overline{\Lambda}(\tau)) \ge L(\overline{x}(\tau), \overline{\Lambda}(\tau)) - \sum_{j=1}^{n} R(\tau, \overline{\xi}_j(\tau)) \ge F^* - n \max_{j=[1,n]} R(\tau, \overline{\xi}_j(\tau)). \tag{3.24}$$

In the case of proper problems, due to Theorem 3.5, the values of all components of the vector functions $\overline{x}(\tau)$ and $\overline{\Lambda}(\tau)$ are bounded on the saddle trajectory. Then, Corollary 3.1 and Theorem 3.3 imply that

$$\lim_{\tau\to+0} \max_{i=[1,m]} R(\tau, \overline{\lambda}_i(\tau)) = \lim_{\tau\to+0} \max_{j=[1,n]} R(\tau, \overline{\xi}_j(\tau)) = 0,$$

and that there exist the limits

$$\lim_{\tau\to+0} U(\tau, \overline{x}(\tau), \overline{\Lambda}(\tau)) = F^* = G^* \quad \text{and} \quad \lim_{\tau\to+0} L(\overline{x}(\tau), \overline{\Lambda}(\tau)) = F^* = G^*. \tag{3.25}$$

Table 2. Solutions to system (3.26) in Example 1 with the parameter $v = 3$

τ	$\bar{\xi}_1(\tau)$	$\bar{\xi}_2(\tau)$	$F(\bar{x}(\tau))$	$\bar{\lambda}_1(\tau)$	$\bar{\lambda}_2(\tau)$	$G(\bar{\Lambda}(\tau))$	$g_1(\bar{\Lambda}(\tau))$
10^{-1}	2.754765504	0.146829492	5.950019486	1.595322348	0.142544871	5.641236270	–0.119587910
10^{-2}	2.980995393	0.011993961	5.997972668	1.615620231	0.185576042	5.960316945	–0.013227685
10^{-3}	2.998148668	1.1754×10^{-3}	5.999823636	1.617360455	0.190653620	5.996003086	-1.3323×10^{-3}
10^{-4}	2.999815350	1.1731×10^{-4}	5.999823636	1.617531210	0.191167734	5.999600031	-1.3332×10^{-4}
10^{-5}	2.999981540	1.1728×10^{-5}	5.999998265	1.617548252	0.191167734	5.999960000	-1.3333×10^{-5}
10^{-6}	2.999998154	1.1728×10^{-6}	5.999999827	1.617549956	0.191224355	5.999996000	-1.3333×10^{-6}

Table 3. Solutions to system (3.26) in Example 1 with the parameter $v = -3$

τ	$\bar{\xi}_1(\tau)$	$\bar{\xi}_2(\tau)$	$F(\bar{x}(\tau))$	$\bar{\lambda}_1(\tau)$	$\bar{\lambda}_2(\tau)$	$G(\bar{\Lambda}(\tau))$	$\bar{F}(x) - \bar{G}(\Lambda)$
10^3	0.999000478	0.999990964	4.997973847	1.006016976	0.997002498	2.963964059	2.034009788
10^2	0.990028539	0.999064507	4.977250598	1.061672873	0.970247397	2.636465765	2.340784833
10	0.890763029	0.891130932	4.454918853	1.717012067	0.721168975	–0.824022353	5.278941206
1	0.104551523	0.048898425	0.355798321	6.557200845	0.086427254	–19.153039010	19.508837331
10^{-1}	8.6106×10^{-4}	4.2695×10^{-4}	3.0030×10^{-3}	60.050951736	8.3357×10^{-3}	–180.102840769	180.105843742
10^{-2}	8.3611×10^{-6}	4.1771×10^{-6}	4.9253×10^{-5}	600.005009704	8.3334×10^{-4}	–1800.010029000	1.8000×10^3
10^{-3}	8.3361×10^{-8}	4.1677×10^{-8}	2.9175×10^{-7}	6.0000×10^3	8.3333×10^{-5}	–18000.001000000	1.8000×10^4

Under the assumptions made above, the vector functions $\bar{x}(\tau)$ and $\bar{\Lambda}(\tau)$ are continuous $\forall \tau > 0$. Therefore, we conclude from the continuity of $L(x, \Lambda)$, inequalities (3.23), (3.24), and the properties of superposition of continuous functions that

$$\lim_{\tau \to +0} L(\bar{x}(\tau), \bar{\Lambda}(\tau)) = L(\lim_{\tau \to +0} \bar{x}(\tau), \lim_{\tau \to +0} \bar{\lambda}(\tau)) = F^* = G^*.$$

In turn, this implies that limits (3.21) exist and satisfy the equalities

$$F(\lim_{\tau \to +0} \bar{x}(\tau)) = F^* \quad \text{and} \quad G(\lim_{\tau \to +0} \bar{\Lambda}(\tau)) = G^*.$$

Finally, in the regular case, the Lagrangian function of problems (2.1) and (2.2) has a unique saddle point $\{x^*, \Lambda^*\}$; therefore, equalities (3.22) hold.

To complete the description of the properties of the U-function for linear problems, we note that the uniqueness of the saddle point of the U-function allows us to consider its representation as a sum of the Lagrangian function and the term

$$\sum_{i=1}^{m} R(\tau, \lambda_i) - \sum_{j=1}^{n} R(\tau, \xi_j)$$

as a regularization method for proper (but irregular) problems, and the U-function itself as a version of the modified Lagrangian function. It is important that the nonnegativity of the bounds on the Lagrange multipliers is guaranteed by conditions (2.4).

Finally, note that feedback functions can also be used for solving nonlinear problems. A discussion of various aspects of such applications and examples can be found in [7].

We illustrate the assertions of Theorems 3.5–3.8 using the following versions of Example 1.

Let the parameter $v = 3$ in Example 1; then the solution is $\xi_1^* = 3$, $\xi_2^* = 0$, $F^* = 6$, and $\lambda_1^* = 2 - 2t$, $\lambda_2^* = t \ \forall t \in \left[0, \frac{1}{3}\right]$, $G^* = 6$. The solution to the primal problem is unique and *overdetermined* (i.e., the number of active constraints at the solution point is greater than the number of variables), and the solution to the dual problem is *not unique*.

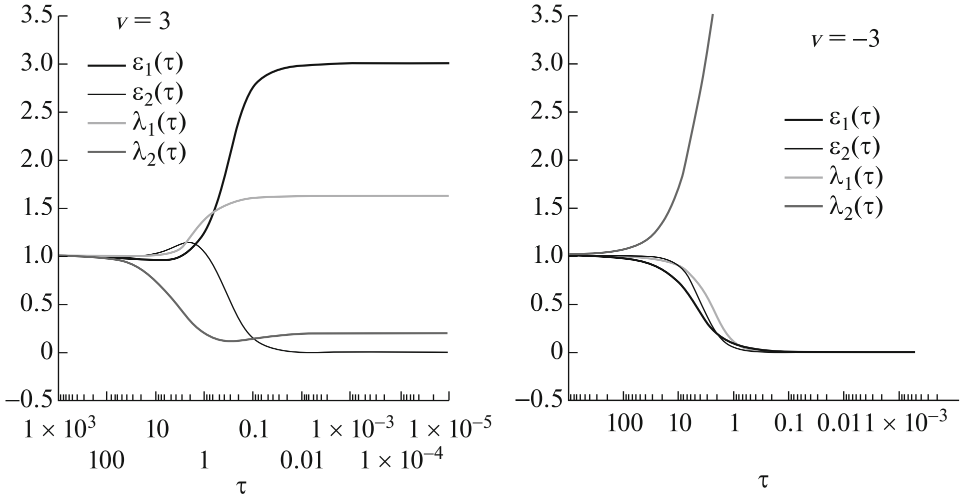

Fig. 1. Plots of the functions $\overline{\xi}_1(\tau)$, $\overline{\xi}_2(\tau)$, $\overline{\lambda}_1(\tau)$, and $\overline{\lambda}_2(\tau)$ for the solutions to system (3.26) with $v = 3$ and $v = -3$.

In Example 1 with the parameter $v = -3$, the primal problem is inconsistent and the dual problem has an *unbounded* solution.

When solving these two versions of Example 1, we use the function $P(\tau, s)$ for which $\dfrac{\partial P}{\partial s} = \dfrac{s}{\tau} + \sqrt{\left(\dfrac{s}{\tau}\right)^2 + 1}$ and, respectively, $Q(\tau, s) = \dfrac{\tau}{2}\left(s - \dfrac{1}{s}\right)$ as an alternative to system (2.8).

In this case, $P(\tau, s)$ is an infinitely differentiable approximation of the standard quadratic penalty function because $\dfrac{s}{\tau} + \sqrt{\left(\dfrac{s}{\tau}\right)^2 + 1} \sim \dfrac{s + |s|}{\tau}$ for small positive τ. In this case, system (2.8) takes the form

$$\begin{aligned}
-3 + \overline{\xi}_1 + 2\overline{\xi}_2 &= \frac{\tau}{2}\left(\overline{\lambda}_1 - \frac{1}{\overline{\lambda}_1}\right),\\
-6 + 2\overline{\xi}_1 + \overline{\xi}_2 &= \frac{\tau}{2}\left(\overline{\lambda}_2 - \frac{1}{\overline{\lambda}_2}\right),\\
-2 + \overline{\lambda}_1 + 2\overline{\lambda}_2 &= -\frac{\tau}{2}\left(\overline{\xi}_1 - \frac{1}{\overline{\xi}_1}\right),\\
-3 + 2\overline{\lambda}_1 + \overline{\lambda}_2 &= -\frac{\tau}{2}\left(\overline{\xi}_2 - \frac{1}{\overline{\xi}_2}\right).
\end{aligned} \tag{3.26}$$

The solutions to system (3.26) for various values of the parameter τ are shown in Tables 2 and 3, and their graphical representation on the logarithmic scale of the argument is shown in Fig. 1. In the tables, we use the notation $\overline{F}(\tau) = F(\overline{\xi}_1(\tau), \overline{\xi}_2(\tau))$, $\overline{G}(\tau) = G(\overline{\lambda}_1(\tau), \overline{\lambda}_2(\tau))$, and $\overline{g}_1(\tau) = g_1(\overline{\lambda}_1(\tau), \overline{\lambda}_2(\tau))$.

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