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REFERENCE MATERIALS OF THE COURSE ELEMENTARY MATHEMATICS

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Some useful information from the elementary mathematics course

It is generally assumed that students initially have a basic knowledge of elementary mathematics as part of the high school curriculum. Nevertheless, it seems appropriate to provide (in reference form) a list of some basic information used in the study of higher mathematics.

1°. **Numbers and their types**

As is well known, the main object of study in mathematics is *numbers,* for a set of which, called *a number set,* operations such as comparison, addition, multiplication, etc., can be performed.

Numbers are divided into

– *natural,* arising when counting objects,

– *integer,* the set of which consists of natural «signed» numbers and the number «zero»,

– *rational,* representable as an irreducible ratio of two integers, and

– *irrational,* whose representation is an infinite non-repeating decimal fraction.

The term *real numbers* is used as a general term for rational and irrational numbers.

2°. Formulas for abbreviated multiplication

For any numbers a and b, the following equalities hold:

$$\begin{aligned}
&(a+b)^2 = a^2 + 2ab + b^2,\\
&(a-b)^2 = a^2 - 2ab + b^2,\\
&a^2 - b^2 = (a+b)(a-b),\\
&(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,\\
&(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3,\\
&a^3 + b^3 = (a+b)(a^2 - ab + b^2),\\
&a^3 - b^3 = (a-b)(a^2 + ab + b^2).
\end{aligned}$$

Recall also that for any *non-negative* numbers a and b, the relation $a + b \geq 2\sqrt{ab}$ holds, and by the definition of the arithmetic square root, it is assumed that

$$\sqrt{a^2} = |a| = \begin{cases} a, & \text{if } a \geq 0,\\ -a, & \text{if } a < 0.\end{cases}$$

Note that $|a|$ is the *absolute value* of a, also known as the *modulus* of a.

3°. Linear Equations

An equation of the form

$$ax + b = 0,$$

where x is the unknown, and $a \neq 0$ and b are fixed numbers, is called the *linear equation.* It has a unique solution: $x = -\dfrac{b}{a}$.

4°. Quadratic Equations

An equation of the form

$$ax^2 + bx + c = 0,$$

where x is the unknown, and $a \neq 0,\ b$ and c are fixed numbers, is called the *square equation.*

For $D > 0$, it has *two* solutions

$$x_1 = \frac{-b + \sqrt{D}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{D}}{2a},$$

where the number $D = b^2 - 4ac$ is called the *discriminant.*

When $D = 0$, the quadratic equation has *one* solution:

$$x = -\frac{b}{2a},$$

and when $D < 0$, it *has no* solutions.

If the quadratic equation has roots, then the following equalities will hold (*Vieta's theorem*):

$$x_1 + x_2 = -\frac{b}{a} \qquad \text{and} \qquad x_1 \cdot x_2 = \frac{c}{a}.$$

5°. **Powers and their properties**

A power of a number a *of order* k (where $k \geq 2$ is a natural number), denoted by a^k, is a product of the form $\underbrace{a \cdot a \cdot a \cdot \ldots \cdot a \cdot a}_{k \text{ factors}}$. In this case, a is called the *base*, and k is the *exponent* of the power.

Powers with a natural exponent $k \geq 2$ have obvious properties:

$$\begin{aligned} &1)\ a^{n+m} = a^n \cdot a^m; \\ &2)\ (a^n)^m = a^{nm}. \end{aligned}$$

The concept of a power of a positive number a can be generalized to the case when its exponent is a rational number of the form $p = \dfrac{m}{n}$, that is, the numbers m and $n \neq 0$ are any integers. To this end (*by definition*), it is assumed that for any $a > 0$ and $a \neq 1$, the following equalities hold:

$$a^1 = a\,; \qquad a^0 = 1\,; \qquad a^{-m} = \frac{1}{a^m}\,; \qquad a^{\frac{1}{m}} = \sqrt[m]{a}.$$

Then, properties 1) and 2) will also hold for powers with a rational exponent:

$$\begin{aligned} &1)\ a^{p+q} = a^p \cdot a^q; \\ &2)\ (a^p)^q = a^{pq}. \end{aligned}$$

In the course on mathematical analysis, it is shown that relations 1) and 2) are also valid for any real numbers p and q, for any positive real number a.

The function $y = a^x,\ a > 0,\ a \neq 1$ is called *exponential*. Fig. 1 shows its graph for $a = 3$.

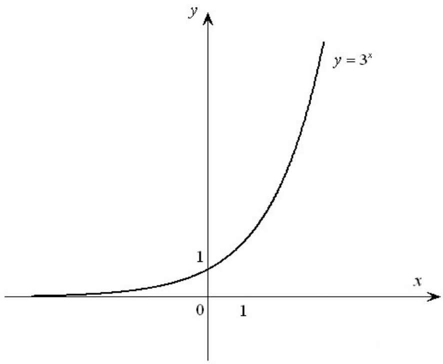

Fig. 1. Graph of an exponential function

6°. **Logarithms and their properties**

The logarithm of a positive number a to the base b (b is a positive number and $b \neq 1$), denoted $\log_b a$, is the *exponent* to which b must be raised to obtain a. The number b is usually called the *base* of the logarithm.

Note that this definition can also be represented as the formula

$$b^{\log_b a} = a, \quad a > 0, \quad b > 0, \quad b \neq 1,$$

which is called the *fundamental logarithmic identity.*

For *decimal* logarithms (that is, logarithms to the base 10), the special notation $\log_{10} a \equiv \lg a$ is used to simplify notation. For the same reason, in higher mathematics, logarithms to the base e (the irrational number $e \approx 2.72\ldots$), called *natural*, are denoted $\log_e a \equiv \ln a$.

Logarithms for any positive numbers a, b, c and $c \neq 1$ have the following properties, following from their definition:

1) $\log_c ab = \log_c a + \log_c b$;

2) $\log_c \dfrac{a}{b} = \log_c a - \log_c b$;

3) $\log_c a^b = b \log_c a$;

4) $\log_c a = \dfrac{\log_b a}{\log_b c}, \quad b \neq 1.$

Formula 4) can be used to convert from one logarithm base to another. For example, $\log_2 17 = \dfrac{\lg 17}{\lg 2}$.

Function $y = \log_a x,\ a > 0,\ a \neq 1$ is called *logarithmic*. Fig. 2 shows its graph for $a = 3$.

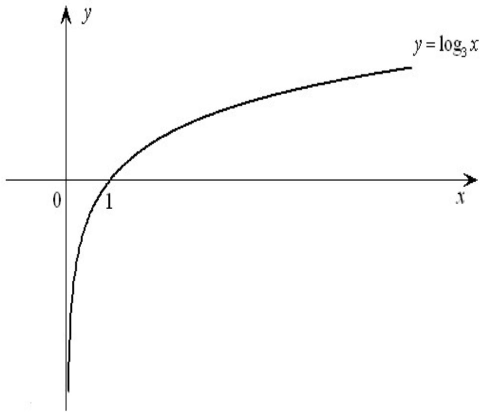

Fig. 2. Graph of a logarithmic function

7°. Trigonometric functions, identities, and equations

Recall that angles can be measured in both *degrees* and *radians.*

One *degree* is the value of the central angle subtended by an arc in a circle whose length is equal to $\frac{1}{360}$ the length of the circle. One *radian* is the value of the central angle in a circle subtended by an arc whose length is equal to the radius of the circle. Since the circumference of a circle is $2\pi r$, an angle of 360^o has a value of 2π radians.

It is convenient to define the basic trigonometric functions using the so-called *trigonometric circle* shown in Fig. 3.

> *The sine of angle* α is the ratio of the y-coordinate of point A to the length of the vector $\overrightarrow{r} = \overrightarrow{OA}$.
> *The cosine of angle* α is the ratio of the x-coordinate of point A to the length of the vector $\overrightarrow{r} = \overrightarrow{OA}$.
> *The tangent of an angle* α is the ratio of the y-coordinate of a point A to its x-coordinate.

Note that by these definitions

1) the sine and cosine have values (not exceeding 1 in absolute value) for *any* angles α. While the tangent can take any value from $-\infty$ to $+\infty$, it does not exist for angles equal to $\frac{\pi}{2} + \pi n$, where n is any integer, that is, for angles $\pm 90^o$, $\pm 270^o$, $\pm 450^o, \ldots$,

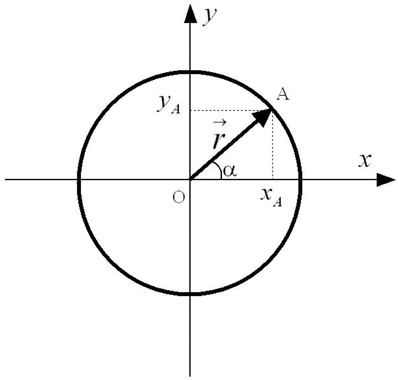

Fig. 3. Definition of Basic Trigonometric Functions

2) trigonometric functions have the property of *periodicity*, that is, their values repeat when the argument α changes by the same minimum possible positive number, called the *period.* The period of the sine and cosine functions is 2π , and that of the tangent is π .

Trigonometric functions of a real argument x (usually measured in radians) are usually denoted as $y = \sin x$, $y = \cos x$ and $y = \operatorname{tg} x$. Their graphs are shown in Fig. 4-6.

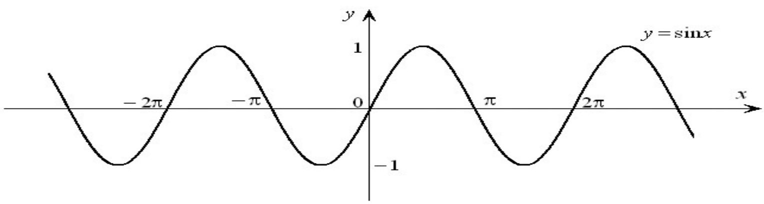

Fig. 4. Graph of the *sine* function

The functions *cotangent, secant* and *cosecant* are also sometimes used.
$\operatorname{ctg} x = \dfrac{1}{\operatorname{tg} x}$, $\sec x = \dfrac{1}{\cos x}$, $\operatorname{cosec} x = \dfrac{1}{\sin x}$.

The following relations hold for trigonometric functions.

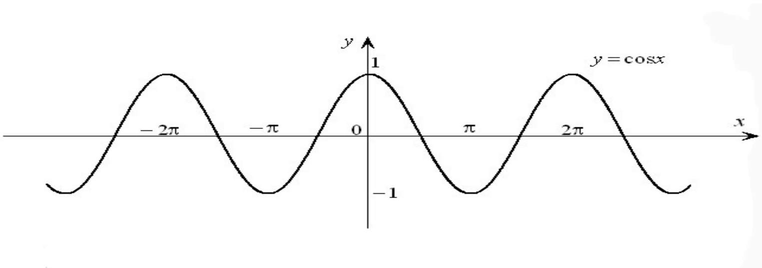

Fig. 5. Graph of the *cosine* function

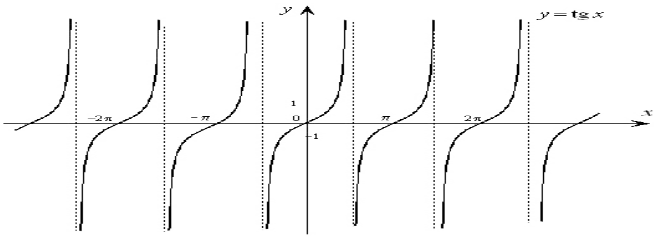

Fig. 6. Graph of the *tangent* function

1) Basic trigonometric identities

$$\sin^2 x + \cos^2 x = 1 \ , \quad \operatorname{tg} x = \frac{\sin x}{\cos x} \ , \quad \frac{1}{\cos^2 x} = 1 + \operatorname{tg}^2 x \ .$$

2) Formulas for the sum (difference) of arguments

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \ ,$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \ ,$$

$$\operatorname{tg}(\alpha \pm \beta) = \frac{\operatorname{tg}\alpha \pm \operatorname{tg}\beta}{1 \mp \operatorname{tg}\alpha\operatorname{tg}\beta} \ ,$$

3) Double argument formulas

$$\sin 2\alpha = 2\sin\alpha\cos\alpha \ , \quad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha \ ,$$

$$\operatorname{tg} 2\alpha = \frac{2\operatorname{tg}\alpha}{1 - \operatorname{tg}^2\alpha} \ .$$

4) Formulas for converting the sum (difference) of trigonometric functions into a product and vice versa

$$\sin\alpha \pm \sin\beta = 2\sin\frac{\alpha\pm\beta}{2}\cos\frac{\alpha\mp\beta}{2}\,,$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\,,$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\,,$$

$$\operatorname{tg}\alpha \pm \operatorname{tg}\beta = \frac{\sin(\alpha\pm\beta)}{\cos\alpha\cos\beta}\,,$$

$$\sin\alpha\sin\beta = \frac{1}{2}\Big(\cos(\alpha-\beta) - \cos(\alpha+\beta)\Big)\,,$$

$$\cos\alpha\cos\beta = \frac{1}{2}\Big(\cos(\alpha-\beta) + \cos(\alpha+\beta)\Big)\,,$$

$$\sin\alpha\cos\beta = \frac{1}{2}\Big(\sin(\alpha+\beta) + \sin(\alpha-\beta)\Big)\,.$$

5) Relations of the form

$$\sin\alpha = \frac{2\operatorname{tg}\dfrac{\alpha}{2}}{1+\operatorname{tg}^2\dfrac{\alpha}{2}}\,, \qquad \cos\alpha = \frac{1-\operatorname{tg}^2\dfrac{\alpha}{2}}{1+\operatorname{tg}^2\dfrac{\alpha}{2}}\,,$$

$$\operatorname{tg}\alpha = \frac{2\operatorname{tg}\dfrac{\alpha}{2}}{1-\operatorname{tg}^2\dfrac{\alpha}{2}}\,,$$

$$\sin 3\alpha = 3\cos^2\alpha\sin\alpha - \sin^3\alpha\,,$$

$$\cos 3\alpha = \cos^3\alpha - 3\sin^2\alpha\cos\alpha\,.$$

Finally, in the case where $\alpha+\beta+\gamma=\pi$ (for example, if α,β,γ are the angles of a triangle) the following equalities will be true:

$$\sin\alpha + \sin\beta + \sin\gamma = 4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\,,$$

$$\operatorname{tg}\alpha + \operatorname{tg}\beta + \operatorname{tg}\gamma = \operatorname{tg}\alpha\cdot\operatorname{tg}\beta\cdot\operatorname{tg}\gamma\,.$$

In computing practice, the problem of determining the size of an angle from the value of one of its trigonometric functions often arises.

To solve this problem, *inverse trigonometric functions* are used: *arcsine* $y = \arcsin x$, *arccosine* $y = \arccos x$ and *arctangent* $y = \operatorname{arctg} x$.

Let's recall the definitions of these functions.

> *arcsine* x , provided that $|x| \le 1$, is a number y such that $|y| \le \dfrac{\pi}{2}$ and $\sin y = x$.
>
> *The arccosine of* x , given that $|x| \le 1$, is a number y such that $0 \le y \le \pi$ and $\cos y = x$.
>
> *The arctangent of* x (for any x) is a number y such that $|y| \le \dfrac{\pi}{2}$ and $\operatorname{tg} y = x$.

When solving applied problems, the formula $\arcsin x + \arccos x = \dfrac{\pi}{2}$ is often useful (easily verified using the definitions given above).

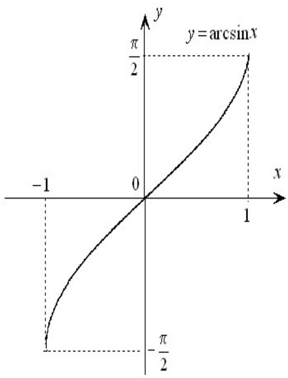

Fig. 7. Graph of the *arcsine* function

Note that the arguments of inverse trigonometric functions are dimensionless, while their values are angles. Typically they are measured in radians.

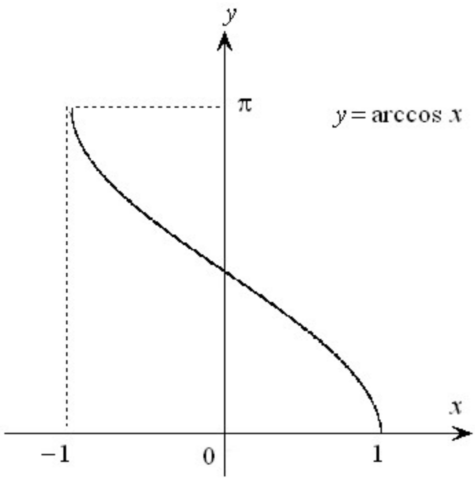

Fig. 8. Graph of the *arccosine* function

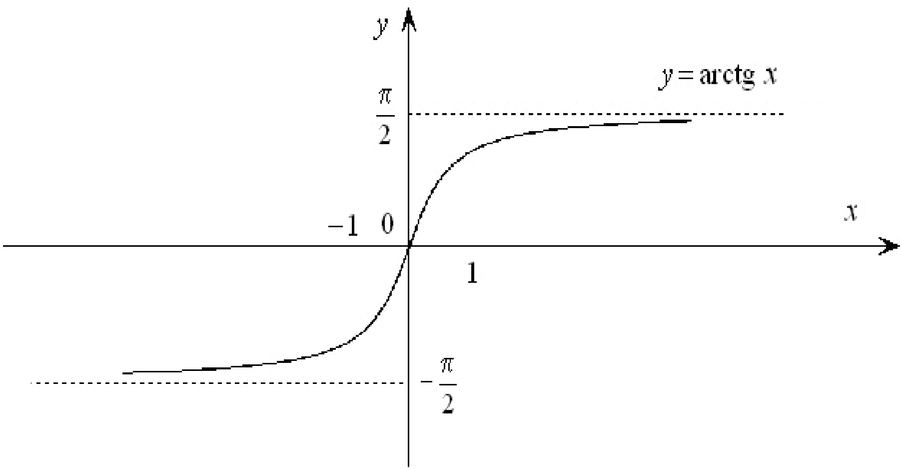

Fig. 9. Graph of the *arctangent* function

Graphs of inverse trigonometric functions are shown in Fig. 7-9.

Inverses trigonometric functions can be used to solve trigonometric equations. For example, the equation

$\sin x = a$ for $|a| \le 1$ has roots of the form $x = (-1)^n \arcsin a + \pi n$,

equation $\cos x = a$ for $|a| \le 1$ has roots $x = \pm \arccos a + 2\pi n$,

equation $\operatorname{tg} x = a$ for any a has roots $x = \operatorname{arctg} a + \pi n$.

In all these formulas n is any integer, that is, $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

8°. **Sets. Elements of Combinatorics**

In mathematics, a *set* is understood to be a collection of objects (or elements) that can be distinguished both from each other and from objects not included in this collection.

The fact that an object x belongs to a set X is usually denoted by $x \in X$. If an object x does not belong to a set X, then $x \notin X$ is used. To denote an *empty* set, that is, one that does not contain any objects, the symbol $\emptyset$ is used. Finally, two sets X and Y consisting of the same objects are said to be *equal,* with equality denoted by $X = Y$.

For sets, there are operations of *union*, denoted by the symbol $\bigcup$, and *intersection*, denoted by $\bigcap$. The notation $x \in X \bigcup Y$ means that the object x belongs to either the set X or the set Y, or both simultaneously. Conversely, the notation $x \in X \bigcap Y$ means that the object x belongs to both the set X and the set Y.

If the objects that form the set X are numbers, then such a set is usually called *numeric.* A set consisting of numbers x satisfying the inequalities $a \le x \le b$ is called a *segment* and is denoted by $[a, b]$. If a numerical set consists of numbers for which $a < x < b$, then it is called an *interval* and is denoted by (a, b).

Finally, the term *interval* denotes either a segment, an interval, or a *half-interval* of the form $(a, b]$ or $[a, b)$. An interval containing a point x is usually called a *neighborhood* of this point.

Let there be a set consisting of n elements. Each *ordered* selection from this set, containing k elements, is called a *placement* of n elements by k elements (sometimes called an "arrangement of n by k elements"). In the case under consideration, it is obvious that $0 \le k \le n$.

The number of all arrangements from n to k is denoted by A_n^k.

Considering that for $k = 0$ there is only one arrangement—the empty set $\emptyset$, then the equality holds:

$$A_n^k = \begin{cases} 1, & \text{if } k = 0, \\ n(n-1)(n-2)\dots(n-k+1), & \text{if } k > 0. \end{cases}$$

The product of all natural numbers from 1 to n is called the *n-factorial*, it is usually denoted by

$$1 \cdot 2 \cdot 3 \cdot \ldots \cdot n = n!$$

(read «*n-factorial*»). Using this notation, the formula for the total number of arrangements is simplified and takes the form

$$A_n^k = \frac{n!}{(n-k)!}.$$

Note that since $A_0^0 = 1$, it makes sense to assume (*by definition*) $0! = 1$. Then this formula will be true for any $0 \le k \le n$.

An arrangement of n elements by n elements is called a *permutation* of n elements. The number of all permutations of n elements is $P_n = n!$.

By definition, permutations can differ from each other both in the composition of their elements and in their order.

If the order of elements in a sample is unimportant, then such a sample of k elements from n is usually called a *combination* of n elements by k elements (sometimes called a "combination of n by k"). The formula for C_n^k – the number of all permutations of n by k – is

$$C_n^k = \frac{n!}{(n-k)!k!}.$$

Using this formula, we can derive the following useful relations: $C_n^k = C_n^{n-k}$ and $(a+b)^n =$

$$= C_n^0 a^n + C_n^1 a^{n-1} b + C_n^2 a^{n-2} b^2 + \dots + C_n^k a^{n-k} b^k + \dots + C_n^{n-1} a b^{n-1} + C_n^n b^n.$$

The last equality is called *Newton's binomial formula* and is a generalization of some abbreviated multiplication formulas given in 2°.

11°. **Useful Inequalities**

For any two non-negative numbers a and b, the following *Cauchy inequality* holds:

$$\frac{a+b}{2} \ge \sqrt{ab}\,.$$

Sometimes it is used in the following form: for any real numbers x and y

$$x^2 + y^2 \geq 2|xy| \ .$$

Cauchy's inequality also holds for a larger number of non-negative integers:

$$\frac{a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n}{n} \geq \sqrt[n]{a_1 a_2 a_3 \cdots a_{n-1} a_n} \ .$$

Note that using the summation and multiplication symbols, the last relation can be written as

$$\frac{1}{n} \sum_{k=1}^{n} a_k \geq \sqrt[n]{\prod_{k=1}^{n} a_k} \ .$$

In many applied problems, the necessary estimates can be obtained using the Bernoulli inequality, which follows from Newton's binomial formula and is valid for any x and any $a > -1$:

$$(1 + a)^x \geq 1 + xa \ .$$

Notes on the Role of the Precision of Definitions and Formulations

In the study of mathematics, special attention should be paid to the completeness and precision of *definitions, theorem formulations and descriptions of properties.* Both redundancy (excessive verbosity) of such lexemes and the loss of any of their details are unacceptable.

We illustrate this with the following examples.

1°. *Arithmetic square root.* As already noted, by definition of the arithmetic square root, it is assumed that $\sqrt{a^2} = |a|$. A question may arise: "Isn't it simpler to assume that $\sqrt{a^2} = a$?"

To demonstrate the incorrectness of this definition, consider the following chain of transformations:

for *any* pair of numbers x and y, the following equalities are true:

$$x^2 - 2xy + y^2 = y^2 - 2yx + x^2,$$
$$(x-y)^2 = (y-x)^2,$$
$$\sqrt{(x-y)^2} = \sqrt{(y-x)^2}.$$

If we now apply a definition of the form $\sqrt{a^2} = a$, we obtain

$$x - y = y - x \quad \Rightarrow \quad x = y,$$

which is obviously $\forall x, y$ is incorrect. While using the definition $\sqrt{a^2} = |a|$ yields

$$|x-y| = |y-x| \quad \Rightarrow \quad 0 = 0,$$

which is true for any pair of numbers x and y.

2°. *How many roots can a quadratic equation have?*

Consider the following three statements A), B) and C):

A) The equation $ax^2 + bx + c = 0$ is a quadratic equation.

B) A quadratic equation cannot have more than two roots.

C) For any pairwise unequal numbers α, β and γ equation

$$\frac{(x-\alpha)(x-\beta)}{(\gamma-\alpha)(\gamma-\beta)} + \frac{(x-\beta)(x-\gamma)}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(x-\gamma)(x-\alpha)}{(\beta-\gamma)(\beta-\alpha)} = 1$$

can be reduced to form A), and it obviously has three different roots $x_1 = \alpha,\ x_2 = \beta,\ x_3 = \gamma$.

It is clear that statements A), B), and C) are contradictory in their entirety. In other words, one of them is erroneous, and at first glance, statement C) raises the greatest doubt. However, it is actually correct, and the error lies in statement A).

The fact is that a quadratic equation is the equation $ax^2 + bx + c = 0$ with $a \neq 0$. And it is for this equation that statement B) is true. In our case, if we reduce equation C) to the form specified in statement A), the coefficient of x^2 will be zero. Moreover, this equation will take the form $1 = 1$, that is, it is an *identity* – a true equality for any value of x (including $x_1 = \alpha,\ x_2 = \beta,\ x_3 = \gamma$).

3°. *Is it possible to arbitrarily group the terms in a sum?* It would seem that the associativity of the addition operation for numbers allows us to give a positive answer to this question. However, this is true only for sums with a *finite* number of terms. If the number of terms in a sum is unlimited, then a situation similar to the following may arise.

Let's accept «on faith» the assertion that the sum of an unlimited number of zeros is zero, and consider a sum of the form

$$A = 1 + 2 - 3 + 1 + 2 - 3 + 1 + 2 - 3 + 1 + 2 - 3 + 1 + 2 - 3 + 1 + 2 - 3 + \ldots$$

by first grouping the terms as

$$A = (1 + 2 - 3) + (1 + 2 - 3) + (1 + 2 - 3) + (1 + 2 - 3) + (1 + 2 - 3) + \ldots$$

We conclude that $A = 0$, since each sum in parentheses is zero.

However, with a different grouping method

$$A = 1 + (2 - 3 + 1) + (2 - 3 + 1) + (2 - 3 + 1) + (2 - 3 + 1) + (2 - 3 + \ldots$$

we obtain $A = 1$. This means that the associative rule is inapplicable to sums with an unlimited number of terms.

The last example clearly demonstrates that «infinity» cannot be treated as an ordinary number.

It is also worth noting that methodologically similar problems can arise in the case of substituting the concepts of «absence of certainty» and «existence of probability», which is often allowed in intuitive reasoning.