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PARAMETRIC SETS OF SECOND-ORDER LINES ON A PLANE

Tutorial manual
on *Analytical Geometry and Linear Algebra*

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The guide presents the theoretical foundations for constructing associations of second-order lines of different types into parametric sets. Examples of using these sets for solving practical problems are given.

Contents

Introduction

When solving any mathematical problem, it is natural to want to simplify the problem statement as much as possible beforehand. However, in some cases, the solution method may consist of generalizing or even complicating it.

One of the methods of this class is *parameterization* of the problem condition, that is, changing its condition by introducing parameters into it in some way.

Let us first clarify the meaning of the concepts used below.

In the framework of this manual, by *parameter* we will mean a mathematical object that is a constant in the problem being solved, the value of which is an element of a certain set.

Let us give an obvious example. The problem *find real solutions to the equation* $x^2 - 6x - 5 = 0$ is parametrically generalized to the form *find real solutions to the equation* $ax^2 + bx + c = 0$, where $a, b, c \in \mathbb{R}$.

It is clear that if we are interested in the roots of only the original equation, then such a complication of the condition is hardly advisable.

Let's consider another example. Let's say we need to find the maximum among the numbers $x_1 = 4, x_2 = -5, x_3 = 0$. Using the exhaustive search obviously gives its solution $x_{max} = 4$. However, it can be obtained (estimate the error on a calculator, for example, at $\tau = 0.1$) by the formula

$$x_{max} = \lim_{\tau \to +0} \tau \ln \left(e^{\frac{x_1}{\tau}} + e^{\frac{x_2}{\tau}} + e^{\frac{x_3}{\tau}} \right) .$$

This formula uses an auxiliary positive parameter τ, by which the limit transition to zero is performed. This formula is, of course, more complex than the program for enumerating answer options, but it does not require logical operations of the type «*if..., then..., otherwise...*»

From the examples given, we can conclude that there are at least two types of parameters:

— *exogenous*, describing the external «information environment» of the task, and

— *instrumental*, not affecting the answer, but necessary for the implementation of the solution search algorithm.

In the examples given, the first type can include the coefficients of a quadratic equation or the values of numbers, among which the maximum is sought. The second type includes an auxiliary parameter τ.

Let's look at another example.
Find the value of the Dirichlet integral

$$I(\alpha)=\int\limits_0^{+\infty}\frac{\sin\alpha x}{x}\,dx,$$

where α is an arbitrary real *exogenous* parameter.

It is impossible to calculate this integral using the Newton-Leibniz formula since the indefinite integral $\int\frac{\sin\alpha x}{x}\,dx$ «not taken», i.e. not represented as some superposition of elementary functions.
However, according to the Dirichlet criterion, this integral converges, i.e. $I(\alpha)$ has a finite value $\forall\alpha\in\mathbb{R}$.

This value can be found by constructing an auxiliary integral

$$\Phi(\alpha,\beta)=\int\limits_0^{+\infty}e^{-\beta x}\,\frac{\sin\alpha x}{x}\,dx,$$

introducing a real *instrumental* parameter $\beta\in[0,1]$.
This integral converges for $\alpha\neq 0$ by the Dirichlet criterion for any fixed $\beta>0$. For $\alpha=0$ it is identically equal to zero.
In this case, the integral of *derivative of the integrand with respect to* α

$$\int\limits_0^{+\infty}e^{-\beta x}\cos\alpha x\,dx$$

will converge by the Weierstrass criterion uniformly on the set $\beta\in(0,1]$ and moreover (this is a theorem!) specifically to $\Phi'_\alpha(\alpha,\beta)$.
In addition, it turns out that the last integral «is taken» by double integration «by parts» and, according to the Newton-Leibniz formula, is equal to (check this yourself) $\dfrac{\beta}{\alpha^2+\beta^2}$.
We have $\Phi(0,\beta)=0$. Then, integrating at a constant value of β $\Phi'_\alpha(\alpha,\beta)=\dfrac{\beta}{\alpha^2+\beta^2}$ over the variable α , we obtain $\Phi(\alpha,\beta)=\operatorname{arctg}\dfrac{\alpha}{\beta}$.

Finally, passing in the last formula to the limit $\beta \to +0$ for a fixed $\alpha > 0$, we obtain

$$I(\alpha) = \lim_{\beta \to +0} \Phi(\alpha, \beta) = \frac{\pi}{2}.$$

And, due to the oddness of the sine, for any α we have $I(\alpha) = \frac{\pi}{2} \operatorname{sgn} \alpha$.

Here it is worth noting that the parameter α in this problem is exogenous, and the parameter β is instrumental.

Thus, based on the examples considered, we can conclude that, although parameterization leads to a formal complication of the problem, the additional degrees of freedom that arise can be used

— both for analyzing the properties of solutions and searching for solutions with special properties,

— and for constructing alternative algorithms for searching for the solutions themselves.

Further in this manual, methods for solving various types of problems based on the parameterization of the description of second-order lines on a plane are considered.

This method is based on the fact that any linear combination of second-order line equations is a line equation of order no higher than 2.

If the desired second-order line must satisfy a certain set of conditions (for example, pass through a given set of points), then it can be assumed that parametrization of a set of such lines will simplify both the formulation of the problem being solved and the method for solving it. For example, for some values of the parameters a linear combination may turn out to be a first-order equation. The reader can find descriptions of the implementation of this idea, for example, in [1,2].

This manual further discusses the conditions for the applicability of this approach and provides examples of solving specific problems.

Parametric sets of second-order lines on the plane

Standard classification of second-order lines on the plane

Second-order lines on the plane are considered in the Cartesian coordinate system, which by default we will consider orthonormal $\{\ O,\ \vec{e}_1,\ \vec{e}_2\ \}$ and we give

Definition 1	If the line L is an algebraic line of the 2nd order, then its equation in the given coordinate system has the form $$Ax^2+2Bxy+Cy^2+2Dx+2Ey+F=0\,, \quad (1)$$ where the numbers A, B, C, D, E and F are any real numbers, and $\vert A\vert+\vert B\vert+\vert C\vert\neq 0$, and x and y are the coordinates of the radius vector of any point belonging to L.

It is obvious that the coefficients of equation (1) for a specific second-order line change when moving from one ONSC to another. Therefore, when studying the properties of these lines, it is advisable to first move to the coordinate system $\left\{\ O',\ \vec{e'}_1,\ \vec{e'}_2\ \right\}$, in which the form of the equation of the line turns out to be *the simplest*.

In the course of analytical geometry, it is proved

Theorem 1 **For any second-order line, there exists an orthonormal coordinate system in which the equation of this line (for $a > 0$, $b > 0$, $p > 0$) has one of the following nine (called *standard*) forms:**

Table 1

Line type $\rightarrow$ $\downarrow$ **Line type**	Elliptic $\Delta > 0$	Hyperbolic $\Delta < 0$	Parabolic $\Delta = 0$
Empty sets	$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = -1$		$y'^2 = -a^2 \quad \forall x'$
Isolated points	$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 0$		
Coincident lines			$y'^2 = 0 \quad \forall x'$
Non-coincident lines		$\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 0$	$y'^2 = a^2 \quad \forall x'$
Curves	*Ellipse* $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$	*Hyperbole* $\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 1$	*Parabola* $y'^2 = 2px'$

where

$$\Delta = \det \left\| \begin{matrix} A & B \\ B & C \end{matrix} \right\| = AC - B^2. \tag{2}$$

We also require that for the standard equation *ellipse* $a \geq b$ **holds.**

To simplify subsequent discussions, we will present this classification in the following form:

Table 2

Line type → ↓ **Cases**	Elliptic $\Delta > 0$	Hyperbolic $\Delta < 0$	Parabolic $\Delta = 0$
Non-degenerate	*Ellipse, imaginary ellipse* $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = \pm 1$	*Hyperbola* $\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 1$	*Parabola* $y'^2 = 2px'$
Degenerate *Imaginary Lines*	$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 0$		$y'^2 = -a^2 \quad \forall x'$
Coincident straight lines			$y'^2 = 0 \quad \forall x'$
Non-coincident straight lines		$\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 0$	$y'^2 = a^2 \quad \forall x'$

Note also that *all* degenerate lines are a pair of real or imaginary lines.

Tables 1 and 2 allow to classify the second-order lines by their standard equations. The proof of Theorem 1, as well as an alternative scheme of parametric classification of the second-order lines defined in the *polar* coordinate system, can be found, for example, in [3].

From Table 2 it follows, which can be verified directly,

`Theorem 2` **For the degeneracy of the second-order line described in Definition 1, it is necessary and sufficient that**

$$\det \left\| \begin{matrix} A & B & D \\ B & C & E \\ D & E & F \end{matrix} \right\| = 0\,.$$

Now let's look at some examples of constructing parametric sets of 2nd order lines.

1) If in equation (1) the coefficient F is taken as a parameter, then such a parametric family will describe the projections of the sections of the surface with the equation $\Phi(x,y,z)=0$ of the form

$$Ax^2+2Bxy+Cy^2+2Dx+2Ey+z=0\,,$$

planes $z=F$ on the coordinate plane Oxy parallel to the applicate axis.

2) A parametric family with parameter λ of the form

$$(A-\lambda)x^2+2Bxy+(C-\lambda)y^2+2Dx+2Ey+F=0$$

consists of 2nd-order lines with symmetry axes parallel to each other. This obviously follows from the formula $\operatorname{ctg}2\varphi=\dfrac{A-C}{2B}$, where φ is the rotation angle necessary to transform the original rectangular coordinate system into the standard one.

3) Let $G_k(x,y)=0 \quad k=\overline{1,n}$ — equations of 2nd order lines of the form (1), having m common points. Then the equation

$$G(x,y)=\sum_{k=1}^{n}\lambda_k G_k(x,y)=0$$

also describes the line passing through these m points, where $\lambda_k \quad k=\overline{1,n}$ not simultaneously equal to zero, real parameters.

Of course, this list is far from exhaustive. Let us just note that the subject of our further consideration will be case 3).

Let us consider example 3) in more detail.

Since in this problem equation (1) has six coefficients to be determined, of which at least one of A, B or C must be nonzero, it is clear that for $n \leq 5$ this problem may have more than one solution, and for the number of points greater than five — it may be unsolvable.

To obtain conditions for the unique solvability of this problem, we will use well-known theorems from the theory of systems of linear equations.

It is easy to see that in the problem under consideration the coefficients of equation (1) must satisfy the following system of linear equations

$$\left\| \begin{array}{cccccc} x_1^2 & 2x_1y_1 & y_1^2 & 2x_1 & 2y_1 & 1 \\ x_2^2 & 2x_2y_2 & y_2^2 & 2x_2 & 2y_2 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x_n^2 & 2x_ny_n & y_n^2 & 2x_n & 2y_n & 1 \end{array} \right\| \left\| \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \end{array} \right\| = \left\| \begin{array}{c} 0 \\ 0 \\ \dots \\ 0 \end{array} \right\| , \qquad (3)$$

where $\{x_k;\, y_k\}$ $k = \overline{1, n}$ — coordinates of given (different!) points.

It is clear that we will be interested only in non-trivial solutions of equation (3).

Consider auxiliary lemma 1.

`Lemma 1` **Through any five non-coinciding points of the plane, three of which belong to the same line, and any four of which do not lie on the same line, it is possible to draw a line of the 2nd order and only one.**

The proof of Lemma 1 is given in Appendix 1.

`Lemma 2` **The n-th equation of system (3) is a linear combination of the first $n-1$ equations if and only if any line passing through the points with coordinates $\{x_1;\, y_1\}, \dots, \{x_{n-1};\, y_{n-1}\}$, necessarily passes through the point with coordinates $\{x_n;\, y_n\}$.**

Prove Lemma 2 yourself.

`Hint`: the assertion of Lemma 2 obviously follows from the fact that, if system (3) contains a dependent equation, then when it is "crossed out", the resulting system is equivalent to the original one.

A corollary of Lemma 2 is Lemma 3.

`Lemma 3` **Prove that for $n \leq 5$ system (3) has linearly dependent equations if and only if at least four points from n lie on one straight line. (If $n < 4$, then the equations in (3) are linearly independent).**

The proof of Lemma 3 is given in Appendix 1.

Now we formulate a generalization of Lemma 1.

`Theorem 3` **Through any five non-coinciding points of the plane, any four of which do not lie on the same line, one can draw a second-order line and only one.**

The proof of Theorem 3 is given in Appendix 1.

We also give

`Definition 2`	The set of *all* lines of the 2nd order passing through a given set of n points, any four of which do not belong to the same line, will be called an *n-point bundle*. For some subset of lines in the bundle, defined in some way, we will use the term *set of lines*.

From Theorem 3 it follows that a 5-point bundle always consists of only one line, while a 6-point or more bundle may be empty.

Let's consider, say, the following example:

$$\begin{array}{lcl} G_1(x,y) & = & x^2 = 0\,; \\ G_2(x,y) & = & y^2 = 0\,; \\ G(x,y) & = & \alpha x^2 + \beta y^2 = 0, \quad \alpha^2+\beta^2 > 0\,. \end{array}$$

It is clear that the lines $G_1(x,y) = 0$ and $G_2(x,y) = 0$ belong to a single-point bundle of lines passing through the origin. At the same time, the parametric set generated by them does not coincide with this bundle. Indeed, the parabola $x^2+y=0$ belongs to the bundle, but is not included in the set.

We will clarify the concept of a set of 2nd order lines by giving

Definition 3

Let a set consisting of n 2nd order lines be given $G_k(x,y) = 0 \quad k = \overline{1,n}$. The set of 2nd-order lines, whose equation has the form

$$\sum_{k=1}^{n} \lambda_k G_k(x,y) = 0, \qquad (4)$$

where $\lambda_k \in \mathbb{R} \;\; k = \overline{1,n}$, is called the *n-parametricset* of 2nd-order lines generated by the set $G_k(x,y) = 0 \quad k = \overline{1,n}$.

Important: set (4), in addition to the equations of the 2nd order lines, contains an equation (called *trivial* for brevity), all coefficients of which are zero, that is, the equation of the coordinate plane Oxy.

This inclusion allows us to establish an isomorphism between the set of equations (4) and the set of 6-component columns of the form

$$\| \, A,\, B,\, C,\, D,\, E,\, F \, \|^{\mathrm{T}}$$

and to consider set (4) as a linear span of the form

$$\left\| \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \end{array} \right\| = \lambda_1 \left\| \begin{array}{c} A_1 \\ B_1 \\ C_1 \\ D_1 \\ E_1 \\ F_1 \end{array} \right\| + \lambda_2 \left\| \begin{array}{c} A_2 \\ B_2 \\ C_2 \\ D_2 \\ E_2 \\ F_2 \end{array} \right\| + \ldots + \lambda_n \left\| \begin{array}{c} A_n \\ B_n \\ C_n \\ D_n \\ E_n \\ F_n \end{array} \right\| .$$

This shell, as is known from the course of linear algebra, is a finite-dimensional subspace, the dimension of which is equal to the maximum number of linearly independent columns in the set

$$\left\{ \| A_k, B_k, C_k, D_k, E_k, F_k \|^{\mathrm{T}} \quad k = \overline{1, n} \right\}.$$

`Important:` equations with proportional coefficients obviously define the same second-order line.

Now let us answer the question: what should a set of second-order lines be like so that it coincides with the bundle in which it is included?

Let us consider the system of linear equations (3) for $n \leq 4$. In the case of $n = 4$, we require that these four points do not belong to the same line.

Let Φ be the fundamental matrix of system (3). In this case we have $\mathrm{rg}\Phi = 6 - n$, and its columns are the coefficients of the fundamental equations of the lines from the bundle.

These equations are linearly independent by virtue of the definition of the fundamental matrix. In this case, the equation of any other line from the bundle can be represented as a linear combination of the fundamental equations. Therefore, system (3) defines the entire bundle.

Thus, in order for a parametric set to coincide with the n-point bundle of which it is a part, it is necessary and sufficient that this set contain $6-n$ lines of the bundle with linearly independent equations.

Thus, in the last of the examples considered, for the set to coincide with the bundle, five lines with linearly independent equations will be required.

On the second-order line passing through four given points in a plane

For Lemma 3 it turns out to be true, useful for applications,

`Corollary 1` **Let $F_k(x, y) = 0 \quad k \in \overline{1, 3}$ are the equations of second-order lines (where the first two are the equations of non-coinciding lines), passing through four given points, any three of which do not lie on the same line. Then $\exists \lambda, \mu \in \mathbb{R}$, such that**

$$F_3(x, y) = \lambda F_1(x, y) + \mu F_2(x, y) = 0\,. \qquad (5)$$

That is, the parametric set (5) will always coincide with a 4-point bundle. Prove it yourself.

Further (for brevity) by a bundle we will mean a 4-point bundle.

The type of line (according to Table 1) is determined by formula (2) by the sign of the quantity Δ.
Let us now analyze which lines are included in the bundle (4). Their type, according to formula (2), is described by the sign expressions

$$\Delta(\lambda,\mu) = \det\left(\lambda \begin{Vmatrix} A_1 & B_1 \\ B_1 & C_1 \end{Vmatrix} + \mu \begin{Vmatrix} A_2 & B_2 \\ B_2 & C_2 \end{Vmatrix}\right) =$$

$$= (\lambda A_1 + \mu A_2)(\lambda C_1 + \mu C_2) - (\lambda B_1 + \mu B_2)^2 =$$

$$= \lambda^2(A_1C_1 - B_1^2) + \lambda\mu(A_1C_2 + A_2C_1 - 2B_1B_2) + \mu^2(A_2C_2 - B_2^2)$$

That is, the function $\Delta(\lambda,\mu)$ is homogeneous in $\{\lambda;\mu\}$, of the second order. Therefore, the bundle (5) can contain no more than two lines of parabolic type, for which $\Delta(\lambda,\mu) = 0$.

`Exercise 1` Prove that if a bundle contains two parabolas, then their axes are never parallel.

We introduce a special notation that will be useful later. Let $A_k \quad k \in \overline{1,4}$ be given distinct points, any three of which do not lie on the same line, and $L_{ij}(x,y) = 0$ is the equation of a line passing through points A_i and $A_j \quad i,j \in \overline{1,4}$.

`Problem 1` *Let P be the intersection point of the altitudes in triangle KMN. Prove that the hyperbolas passing through points K, M, N and P have perpendicular asymptotes.*

`Solution.` According to the problem statement $KM \perp NP$ and $KN \perp MP$, then in the introduced notations we have $L_{KM}L_{NP} = 0$ and $L_{KN}L_{MP} = 0$. Each of these equalities is a 2nd order equation of the form (1), for which $A + C = 0$.

Since by Theorem 4 all the lines of the 2nd order, passing through the points K, M, N and P, have equations of the form:

$$\alpha L_{KM} L_{NP} + \beta L_{KN} L_{MP} = 0\,, \qquad (6)$$

then in (6) $\forall \alpha, \beta$ when reduced to form (1) we also get $A + C = 0$.

Solution is found. Therefore (check it yourself), all lines of this set are of hyperbolic type and with $A + C = 0$, and hyperbolas with $A + C = 0$ have perpendicular asymptotes.

Exercise 2 Consider also the question: is it possible for the bundle to consist only of lines of the 2nd order

a) elliptic type,

b) hyperbolic and parabolic types?

Problem 2 In an orthonormal coordinate system the following points are given: $A_1 = \|\, 3\ 1 \,\|^{\mathrm{T}}$, $A_2 = \|\, -2\ 3 \,\|^{\mathrm{T}}$, $A_3 = \|\, -1\ 0 \,\|^{\mathrm{T}}$ and $A_4 = \|\, 2\ {-2} \,\|^{\mathrm{T}}$. It is required to construct a parametric description of a set of second-order lines passing through these points.

Solution. Let the following points be given in an orthonormal coordinate system: $A_1 = \|\, 3\ 1 \,\|^{\mathrm{T}}$, $A_2 = \|\, -2\ 3 \,\|^{\mathrm{T}}$, $A_3 = \|\, -1\ 0 \,\|^{\mathrm{T}}$ and $A_4 = \|\, 2\ {-2} \,\|^{\mathrm{T}}$, for which the linear functions, specified in the formulation of Theorem 4, have (check this yourself!) the form:

$$\begin{aligned} L_{12}(x,y) &= 2x + 5y - 11, \\ L_{23}(x,y) &= 3x + \ \ y + \ \ 3, \\ L_{34}(x,y) &= 2x + 3y + \ \ 2, \\ L_{41}(x,y) &= 3x - \ \ y - \ \ 8. \end{aligned}$$

Then the parametric representation of the set of 2nd order lines passing through these points will be as follows:

$$\alpha(2x+5y-11)(2x+3y+2)+\beta(3x+y+3)(3x-y-8) = 0.$$

Fig.1 shows graphical representations of some members of this set:

— *in red* shows the ellipse obtained when $\alpha = \beta = 1$;

— *in green — hyperbola with* $\alpha = 1$ *and* $\beta = -3$ *(the green* dashed lines show its asymptotes);

— *in blue* — parabola, for which parameter values $\alpha = \dfrac{131 + \sqrt{17017}}{8}$ and $\beta = 1$;

— *in gray* color — a pair of intersecting lines with $\alpha = -1$ and $\beta = 1$.

If $\beta = 0$, then $\Delta = -4\alpha^2$, and obviously $\Delta < 0$. For $\beta \neq 0$ the resulting trinomial is factored into

$$\Delta = -4\beta^2 \left(\frac{\alpha}{\beta} - k_1 \right) \left(\frac{\alpha}{\beta} - k_2 \right), \qquad (7)$$

where the numbers k_1 and k_2 are the roots of the quadratic equation $k^2 - \dfrac{131}{4}k + \dfrac{9}{4} = 0$, equal respectively to

$$k_1 = \frac{131 + \sqrt{17017}}{8} \approx 32.681;$$

$$k_2 = \frac{131 - \sqrt{17017}}{8} \approx 0.069.$$

In the example under consideration, $A = 4\alpha + 9\beta$, $B = 8\alpha$ and $C = 15\alpha - \beta$, so

$$\Delta = \det \left\| \begin{array}{ll} 4\alpha + 9\beta & 8\alpha \\ 8\alpha & 15\alpha - \beta \end{array} \right\| = -4\alpha^2 + 131\alpha\beta - 9\beta^2.$$

From (7) it follows that we have lines of the 2nd order *parabolic* type, for $\alpha = k_1\beta$ or for $\alpha = k_2\beta$. If the quadrilateral $A_1A_2A_3A_4$ is a trapezoid, then this line is one parabola and *a pair of parallel lines* or two such pairs.

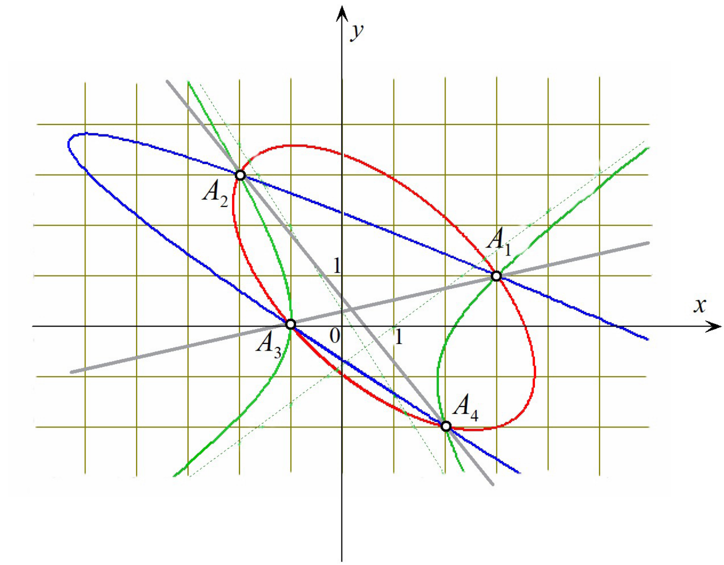

Fig.1. Some 2nd order lines of the parametric set
$\alpha(2x+5y-11)(2x+3y+2)+\beta(3x+y+3)(3x-y-8)=0.$

Otherwise (as it turns out in our example) these are two parabolas. We suggest you figure out the details yourself. If the insides of the parentheses in (7) have different signs, then the line type is — *elliptical*, and the line itself will be *ellipse*. Other types of elliptical type are impossible, since these points do not coincide.

Finally, if the interiors of the brackets have the same signs, then the desired 2nd order line belongs to the *hyperbolic* type.

`Solution is found.` The line will be *a hyperbola*, except for the cases $\alpha\beta = 0$ or $\alpha = -\beta$, when it turns out to be *a pair of intersecting lines*.

Figure 2 graphically shows the dependence of the type and type of the 2nd order line on the values of the parameters α and β in problem 2.

Points on the plane $\{0\alpha\beta\}$ are painted in different colors depending on the type and type of the 2nd order line:

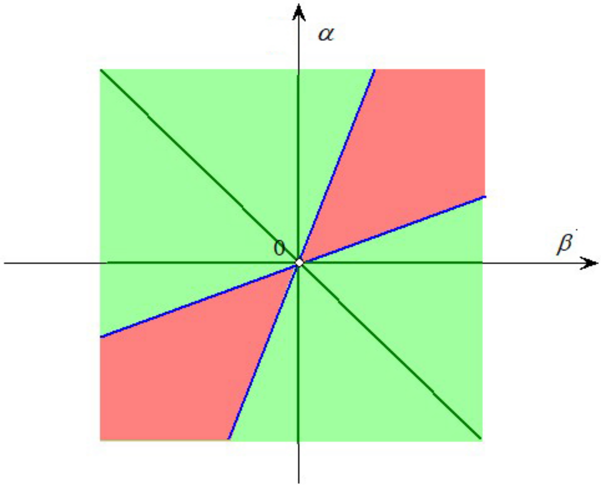

Fig. 2. Dependence of the type and type of the 2nd order line on the values α and β

— *pink* color marks the cases of *ellipses*;

— *blue* color — *parabolic cases*, i.e. points on $\{0\alpha\beta\}$ that lie on the lines: either $\alpha = k_1\beta$, or $\alpha = k_2\beta$;

— *light green* color — cases of *hyperbolas*;

— *green* color — cases of *pairs of intersecting lines* related to hyperbolic type, that is, points on the plane $0\alpha\beta$, which belong to one of the three lines $\beta = 0$, $\alpha = 0$ and $\beta = -\alpha$.

`Problem 3` *Two parabolas whose axes are perpendicular have four points of intersection. Prove that these points lie on the same circle.*

`Solution.` We choose a coordinate system in which the axes of symmetry of the parabolas lie on the coordinate axes, and the equations of the parabolas are as follows:

$$y^2 = 2p(x - x_0) \qquad \text{and} \qquad x^2 = 2q(y - y_0)\,, \tag{8}$$

where $p > 0$, $q > 0$, $x_0 < 0$ and $y_0 < 0$ (see Fig. 3).

If we use Corollary 1 and construct a linear combination of equations (8), which turns out to be the equation of a circle, then the problem will be solved.

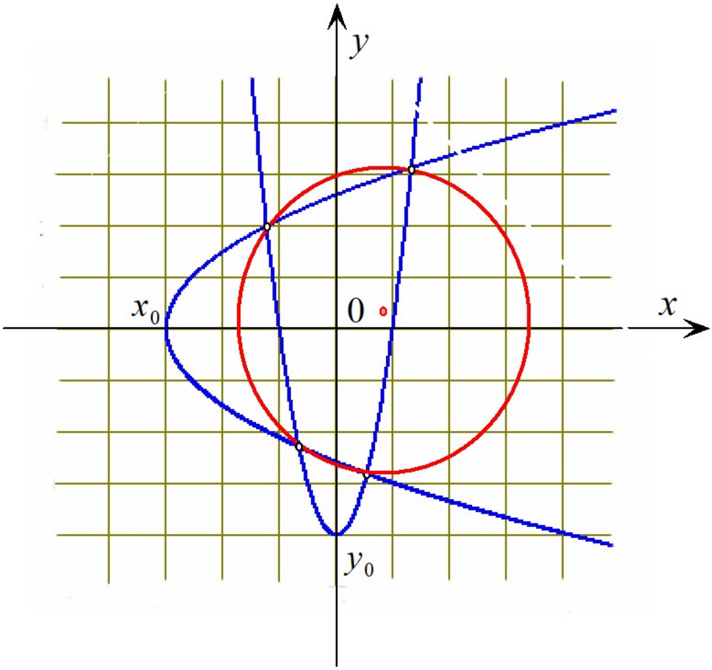

Fig. 3. To solve problem 3

If we use Corollary 1 and construct a linear combination of equations (8), which turns out to be the equation of a circle, then the problem will be solved.

If we put in formula (4) $\lambda = \mu = 1$, then we get

$$\lambda \left(y^2 - 2p(x - x_0)\right) + \mu \left(x^2 - 2q(y - y_0)\right) =$$

$$= x^2 - 2p(x - x_0) + y^2 - 2q(y - y_0) = 0\,.$$

Where from

$$x^2 - 2px + p^2 + y^2 - 2qy + q^2 = -2px_0 + p^2 - 2qy_0 + q^2\,,$$

Solution is found.

which is the equation of a circle in ONSC:

$$(x - p)^2 + (y - q)^2 = \underbrace{-2px_0 - 2qy_0}_{>0} + p^2 + q^2 = R^2\,.$$

`Exercise 3` *Prove a generalization of the statement contained in the conditions of problem 3:*

Let two second-order lines have four common points. These points lie on the same circle if and only if the axes of these lines are perpendicular.

Using this parametric description of a set of second-order lines, by choosing the values of the parameters α and β we can obtain equations of second-order lines with certain geometric properties.

Let us consider other problems that demonstrate the usefulness of parametric sets of 2nd order lines.

`Problem 4` *«Butterfly Theorem»* *Given a circle in which chords AB and CD intersect chord PQ at point O — its midpoint. Prove that the chords AD and CB intersect PQ at points equidistant from O.*

`Solution.` Second-order lines: a circle ω of radius R and two pairs of intersecting lines $f : L_{AB}L_{CD}$ and $g : L_{AT}L_{CT}$ obviously belong to the same set of second-order lines. Therefore, according to Corollary 1, $g = \omega + \lambda f$.

Let us choose a Cartesian coordinate system such that its origin is at the point O (see Fig. 4), and the segment PQ lies on the Ox axis. Then

$$\begin{aligned} \omega(x,y) &= x^2 + (y+y_0)^2 - R^2, \\ f(x,y) &= (x+py)(x+qy), \end{aligned}$$

where p and q are some constants.

Since

$$g(x,y) = \omega(x,y) + \lambda f(x,y)$$

is true for all y, then

$$g(x,0) = 0 \qquad \Longleftrightarrow$$

$$\Longleftrightarrow \qquad x^2 + \lambda(x^2 + y_0^2 - R^2) = 0$$

will also be true.

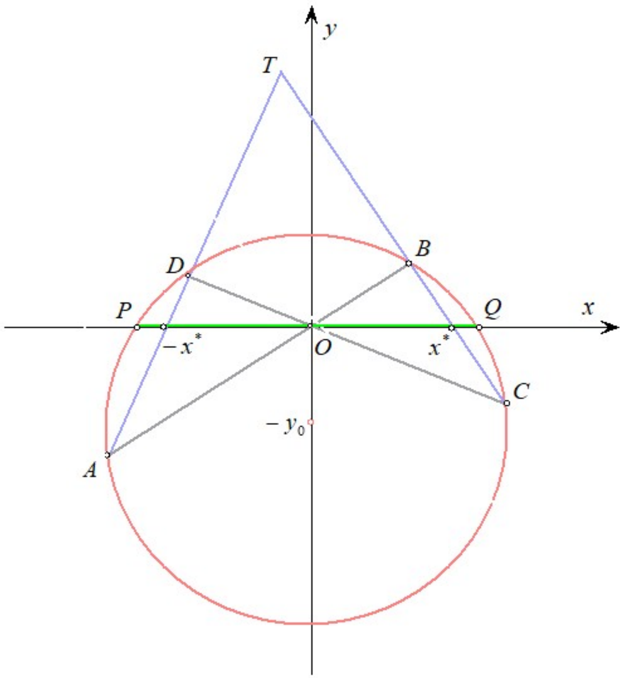

Fig. 4. To the solution of problem 4

The roots of the last equation are the abscissas of the intersection points of the chord PQ with the chords AD and CB.

`Solution is found.` These roots: $\pm x^*$, are equal in absolute value and have different signs, from which follows the validity of the statement being proved.

`Problem 5` *The equations of the diagonals of a square are*

$$\begin{array}{rcrcl} x & - & 8y & = & 38, \\ 8x & + & y & = & 44, \end{array}$$

and the length of its side is $\sqrt{130}$. *Find the equations of the sides of the square.*

`Solution.` 1°. Let us consider the problem in an orthonormal coordinate system, in which the diagonals of the square are on the coordinate axes, and the origin $0'$ is the intersection point of the diagonals (see Fig. 5).

The coordinates of the point $0'$ — the new origin — are found by solving the system of linear equations

$$\begin{cases} x - 8y = 38, \\ 8x + y = 44. \end{cases}$$

We obtain $0'\{6; -4\}$.

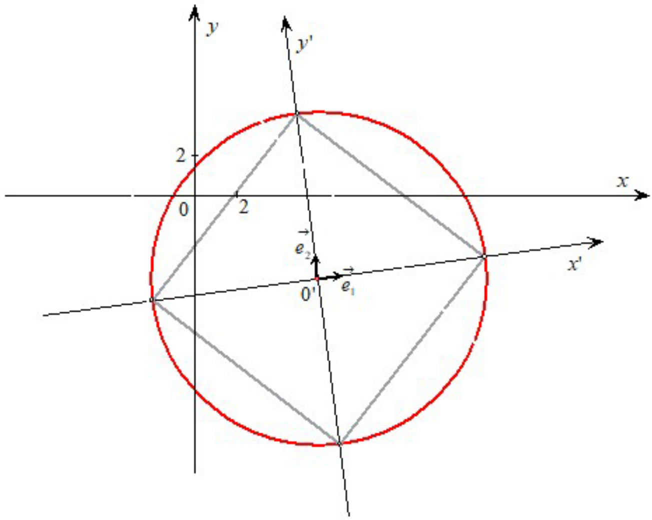

Fig. 5. To solve problem 5

As new basis vectors, we take the normalized direction vectors of the diagonals of the square. Since for the straight line $Ax + By + C = 0$ the vector $\|-B\ A\|^{\mathrm{T}}$, can serve as a guide vector, then we take the vectors $\left\|\vec{e}_1^{\,\prime}\right\| = \left\|\frac{8}{\sqrt{65}}\ \frac{1}{\sqrt{65}}\right\|^{\mathrm{T}}$ and $\left\|\vec{e}_2^{\,\prime}\right\| = \left\|-\frac{1}{\sqrt{65}}\ \frac{8}{\sqrt{65}}\right\|^{\mathrm{T}}$ as the basis vectors.

Therefore, *transition formulas* from the original orthonormal coordinate system to the new one will have the form:

$$\left\{ \begin{array}{rcrcrcr} x & = & \frac{8}{\sqrt{65}}x' & - & \frac{1}{\sqrt{65}}y' & + & 6, \\ y & = & \frac{1}{\sqrt{65}}x' & + & \frac{8}{\sqrt{65}}y' & - & 4. \end{array} \right. \qquad (9)$$

2°. Let us now consider a parametric set of lines of the 2nd order, passing through four points — the vertices of the square.

One of the lines of this set is *a pair of intersecting lines*, on which lie the diagonals of the square. It belongs to the hyperbolic type.

Another is *a circle* of radius $\sqrt{65}$ with center at the intersection point of the diagonals. This is an elliptic type.

Finally, there are two *pairs of parallel lines* on which non-adjacent sides of the square lie. Here the type is parabolic. Note that the goal of the problem is to find the equations of these parallel lines.

In the original coordinate system, the equation of a pair of intersecting lines, on which the diagonals lie, will be

$$(x - 8y - 38)(8x + y - 44) = 0.$$

Check for yourself, that by (9) this equation in the new coordinate system will take the form: $x'y' = 0$.

The equation of the circle passing through the vertices of the square in the new coordinate system is: $x'^2 + y'^2 = 65$. Then, by Corollary 1, the desired equation in the new coordinate system has the form:

$$\lambda(x'^2 + y'^2 - 65) + \mu\, x'y' = 0. \qquad (10)$$

$\lambda = 0$ does not give a solution here, since in this case the equation defines only lines of hyperbolic type.

Therefore, we set in (10) $\lambda = 1$ and find, at what μ it defines lines of parabolic type. Then from the equation

$$\Delta = \det \begin{Vmatrix} 1 & \frac{1}{2}\mu \\ \frac{1}{2}\mu & 1 \end{Vmatrix} = 0 \qquad \Longrightarrow \qquad \mu = \pm 2.$$

This gives the equations

$$(x' + y')^2 = 65 \qquad \text{and} \qquad (x' - y')^2 = 65\,. \qquad (11)$$

And, since the vertices of the square lie on parallel lines, other cases of a parabolic line (parabola or coinciding lines) are impossible here.

3°. Let us now find the form of these equations in the original coordinate system.

Since both coordinate systems are orthonormal, the matrices of the direct and inverse transitions for them are orthogonal. Using this fact, from (9) we obtain

$$\left\{ \begin{array}{rcrcrcr} x' & = & \frac{8}{\sqrt{65}}x & + & \frac{1}{\sqrt{65}}y & - & \frac{44}{\sqrt{65}}, \\ y' & = & -\frac{1}{\sqrt{65}}x & + & \frac{8}{\sqrt{65}}y & + & \frac{38}{\sqrt{65}}. \end{array} \right.$$

Finally, substituting these expressions into (11) yields the desired equations of pairs of parallel lines

$$\left[\begin{array}{rcrcr} 9x & - & 7y & = & 147, \\ 9x & - & 7y & = & 17 \end{array} \right.$$

and

`Solution is found.`

$$\left[\begin{array}{rcrcr} 7x & + & 9y & = & 71, \\ 7x & + & 9y & = & -59. \end{array} \right.$$

`Problem 6` *«Pascal's Theorem»*

If non-coinciding points A_1, A_2, A_3, A_4, A_5, A_6 lie on a 2-fold line ω, then the intersection points (if such exist) of the lines L_{12} and L_{45}, L_{23} and L_{56}, L_{34} and L_{16} lie on the same line.

`Solution.` Each set of four points from six given generates a bundle of lines of the 2nd order, to which the line ω belongs.
Let us select two bundles among them containing the points A_1, A_2, A_3, A_4 and A_6, A_1, A_4, A_5 respectively.
The equations of the line ω in these bundles will be

$$\begin{aligned} \omega : &\quad \alpha_1 L_{12} L_{34} + \beta_1 L_{14} L_{23} = 0\,, \\ \omega : &\quad \alpha_2 L_{16} L_{45} + \beta_2 L_{14} L_{56} = 0\,. \end{aligned}$$

Subtracting these equations term by term, we obtain

$$\alpha_2 L_{16} L_{43} - \alpha_1 L_{12} L_{34} + L_{14}\big(\beta_2 L_{56} - \beta_1 L_{23}\big) = 0\,.$$

If we substitute the coordinates of the point $A^* : \begin{cases} L_{16} = 0\,, \\ L_{34} = 0\,, \end{cases}$ then we get that the point A^* belongs to the line with the equation $\Omega :\ \beta_2 L_{56} - \beta_1 L_{23} = 0$. This is true, since the point A^* does not belong to the line with the equation $L_{14} = 0$, which would be possible only if the points A_1 and A_4 coincide.

Arguing similarly, we get that the point $A^{**} : \begin{cases} L_{15} = 0\,, \\ L_{24} = 0\,, \end{cases}$ also belongs to the line Ω.

Finally, note that the point $A^+ : \begin{cases} L_{23} = 0\,, \\ L_{56} = 0\,, \end{cases}$ belongs to the line Ω by the definition of the function $L_{ij}(x, y)$. Therefore, three points lie on this line: A^*, A^{**} and A^+.

`Solution obtained.`

The line Ω is called *Pascal's line.*

Note also that the condition does not specify whether the closed line $\overline{A_1, A_2, A_3, A_4, A_5, A_6}$ has self-intersection points or not. In this case, different order of numbering of points gives, generally speaking, different Pascal's lines.

Parametric sets of second-order lines can be useful for solving not only geometric problems.

An example of such a case is the problem of choosing a replacement of an unknown that leads to a decrease in the order of the equation being solved.

Let it be required to solve a fourth-order equation

$$x^4 + ax^3 + bx^2 + cx + d = 0\,, \tag{12}$$

where a, b, c, $d \in \mathbb{R}$.

A number of methods for solving equation (12) are currently known. For example, the Ferrari method, which, like other methods, is based on solving the resolvent — auxiliary equation of the 3rd degree.

Let us consider a method for constructing a resolvent using a parametric set of second-order lines.

It is easy to verify that equation (12) and the system of equations

$$\begin{cases} \quad\quad y - x^2 &= 0, \\ y^2 + axy + by + cx + d &= 0 \end{cases} \tag{13}$$

are equivalent.

Let the equations of system (13) be the equations of the 2nd order lines, the left-hand sides of which we denote as f and g, respectively.

To solve system (13), and, consequently, equation (12), means: to find the coordinates of the intersection points of the lines $f = 0$ and $g = 0$. Note that in (13) the second line $g = 0$ can be replaced by the line $\lambda f + g = 0$, where λ is some real parameter. In this case, the solutions (13) will remain the same as before.

However, if in (13) the second line is *degenerate*, then the solution of system (13) is reduced to finding only the roots of some quadratic equations.

The degeneracy condition of the line $\lambda f + g = 0$

$$\lambda\Big(y - x^2\Big) + y^2 + axy + by + cx + d = 0$$

or

$$-\lambda x^2 + axy + y^2 + cx + (b+\lambda)y + d = 0\,,$$

by virtue of Theorem 2 has the form of equality

$$\det \left\| \begin{matrix} -2\lambda & a & c \\ a & 2 & b+\lambda \\ c & b+\lambda & -2d \end{matrix} \right\| = 0\,,$$

which is a cubic equation with respect to the parameter λ.

The resulting equation has the form

$$\lambda^3 + 2b\lambda^2 + (ac + b + 4d)\lambda + da^2 + abc - c^2 = 0 \qquad (14)$$

is the desired resolvent.

Indeed, let λ be a root of (14). In this case $\lambda f + g = 0$ is the equation of a degenerate line of the second order, the left side of which is decomposed into two linear factors. Then, using the equality $y = x^2$ allows us to find the roots of (12) by solving only two quadratic equations.

Appendix 1

Proofs of Lemmas and Theorems

Lemma 1 **Through any five non-coinciding points of the plane, three of which belong to the same line, and any four of which do not lie on the same line, it is possible to draw a line of the 2nd order and only one.**

Proof.

First, let us verify that if four of the five given points lie on the same line, then the second-order line passing through these points is not the only one.

Indeed, let four points out of five lie on the line $ax + by + c = 0$, where the numbers a, b and c are determined uniquely. And let the line $Ax + By + C = 0$ pass through the fifth point, for which there are infinitely many possible values of the numbers A, B and C. Then any second-order line of the form

$$(ax + by + c)(Ax + By + C) = 0$$

passes through all five given points.

Now consider the case when only three of the five points lie on the same line.

Since a non-degenerate line of the 2nd order can have no more than two points of intersection with any line, then the line of the 2nd order under consideration is degenerate.

A degenerate line in the case under consideration is the union of two lines (possibly parallel), one of which passes through three points lying on the same line. And the second passes through the remaining two.

From which follows *uniqueness* of such a line of the 2nd order.

The lemma is proved.

Lemma 3 **Prove that for $n \leq 5$ system (3) has linearly dependent equations if and only if at least four points from n lie on one line. (If $n < 4$, then the equations in (3) are linearly independent).**

Proof.

We will give the proof for the case $n = 4$. The case $n = 5$ is proved similarly.

Sufficiency obviously follows from the fact that, if three different points of one line belong to the line, then this line is degenerate and the line under consideration belongs to it entirely. Then the equations of system (3) corresponding to these points are linearly dependent by virtue of Lemma 2.

Proof of necessity.
If the first three points do not lie on the same line through them one can always draw a pair of intersecting lines and an ellipse. The fact that these two lines have no other common points contradicts Lemma 2.

Now if we assume that the first three points lie on one line, a line of the form «a pair of coinciding lines» passes through them, and, according to Lemma 2, the point $\{x_4; y_4\}$ must lie on it, that is, all four given points lie on one line. Which was to be proved.

The lemma is proved.

The sufficiency statement will also be true for $n > 5$, but the necessity statement will not, due to the dimensionality of the system. As a counterexample, we can take the vertices of a hexagon inscribed in a circle.

Theorem 3 **Through any five non-coinciding points of the plane, any four of which do not lie on the same line, one can draw a line 2nd order and only one.**

Proof.

The case when three of the five points lie on the same line is considered in Lemma 1.

Now let no three of the five points lie on the same line. Then the second-order line passing through them is non-degenerate.

Now consider four points out of five. According to the condition, they are the vertices of a quadrangle with non-empty interior (this quadrangle may not be convex, as, for example, there is it in problem 1).

Therefore, the pairs of lines on which the non-adjacent sides of the quadrilateral lie will be lines of the 2nd order (hyperbolic or parabolic types) passing through these four points.

Let the equations of these lines be: $\Phi_1(x,y) = 0$ and $\Phi_2(x,y) = 0$. Then the line defined by the equation

$$\Phi(x,y) = \alpha\Phi_1(x,y) + \beta\Phi_2(x,y) = 0$$

will be a line of the 2nd order passing through these four points for any α and β that are not simultaneously equal to zero.

For the coordinates of the fifth point $\{x_5,\ y_5\}$ there are always α and β such that

$$\alpha\Phi_1(x_5,y_5) + \beta\Phi_2(x_5,y_5) = 0\,.$$

Moreover, since the point $\{x_5,\ y_5\}$ does not belong to any of the lines connecting any pair of vertices of the resulting quadrilateral, then $\Phi_1(x_5,y_5) \neq 0$ and $\Phi_2(x_5,y_5) \neq 0$.

This proves the existence of a non-degenerate line of the 2nd order, passing through any five points.

Finally, by virtue of the statement of Exercise 1, the proof of which is given below, the rank of the fundamental matrix of system (3) for the indicated points is equal to one. Which proves the uniqueness of the line, passing through these points.

The theorem is proven.

Appendix 2

Solution of exercises

Exercise 1 *Prove that if a bundle contains two parabolas, then their axes are never parallel.*

Solution. Different parabolas from one bundle must have four common points. For parabolas with parallel axes this is obviously not true.

Note that similar reasoning applies when a parabola and a pair of parallel lines, or two pairs of parallel lines, are considered.

Solution obtained.

Exercise 2 *Consider also the question: is it possible for the bundle to consist only of 2nd order lines*

a) elliptic type,

b) hyperbolic and parabolic types?

Solution. a) The answer to this question is negative. Indeed, through any four points one can draw a pair of intersecting lines. Therefore, any bundle contains lines of hyperbolic type.
Moreover, if the bundle contains an ellipse, then by virtue of the continuity of the function $\Delta(\lambda, \mu)$ it will contain (as an intermediate case of elliptic and hyperbolic types) two lines of parabolic type.

b) Here the answer is also negative.
Any quadrilateral whose vertices lie on the boundary of a convex set is also convex. The interiors of a parabola and pairs of parallel lines are convex sets. Therefore, the quadrilateral formed by the common points of the bundle is also convex.

It is also known that an ellipse can be described around any convex quadrilateral.
Indeed, an affine transformation can always ensure that the sum of the opposite interior angles in a quadrilateral is equal to π. And this is a necessary and sufficient condition for the quadrilateral to be inscribed in a circle.

Since the two-parameter set of lines of a four-point bundle coincides with the bundle by virtue of Theorem 4, the indicated ellipse of the bundle also belongs to the set.
This means that if the bundle contains a parabolic line, then it also contains an elliptic line. Therefore, the answer is negative.

`Solution is found.`

Thus, it is clear that for any bundle of 2nd-order lines passing through four given points, only two cases are possible:

1) if the points form a convex, then the bundle consists of a pair of parabolic-type lines and infinite sets of both elliptic and hyperbolic types.

2) in the case of a non-convex quadrangle, the bundle consists only of hyperbolic-type lines.

Note also that a parametric set of 2nd-order lines described by a formula of the form (5) can consist only of hyperbolic-type lines and a single parabolic-type line.

But the lines of this set will not have exactly four common points, and therefore will not be a bundle.

`Exercise 3`

Prove the following generalization of the statement contained in the conditions of problem 3:

Let two lines of the second order have four common points. These points lie on the same circle if and only if the axes of these lines are perpendicular.

Solution. *Sufficiency.* Let the bundle be formed by two lines of the 2nd order with mutually perpendicular axes and have the equation $f = \lambda f_1 + \mu f_2$. Let us pass to a rectangular coordinate system whose axes are parallel to the axes of the lines $f_1 = 0$ and $f_2 = 0$. Obviously, for them the coefficients B in (1) are equal to zero. Then for any line in this bundle $B = 0$.

Now we choose λ and μ so that in the equation of the line $f = 0$ we have $A = C$. This gives

$$\lambda A_1 + \mu A_2 = \lambda C_1 + \mu C_2 \;\Leftrightarrow\; \lambda(A_1 - C_1) + \mu(A_2 - C_2) = 0. \tag{16}$$

For $\mu = 0$ and $\lambda \neq 0$ the line $f_1 = 0$ will be a circle. Similarly, for $\mu \neq 0$ and $\lambda = 0$ the circle will be the line $f_2 = 0$. By virtue of $B = 0$ and (16) the sufficiency is proved.

Necessity. This bundle contains a circle ω. That is, $\exists\, \lambda$ and $\exists \mu$ such that $\omega = \lambda f_1 + \mu f_2 = 0$.

Without loss of generality, we can assume that $f_1 = 0$ is neither a parabola nor a circle. (Show yourself that such a line always exists in the bundle).

Let us choose a rectangular coordinate system in which the axes are perpendicular to the axes of the line $f_1 = 0$. Then $B_1 = 0$, and by virtue of $B_\omega = 0$ we will also have $B_2 = 0$. Thus, the axes of the lines $f_1 = 0$ and $f_2 = 0$ are perpendicular.

Solution obtained.

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